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Relativistic methods in non-commutative quantum mechanics

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Abstrakt

Venujeme sa konštrukcii Diracovho operátora častice so spinom $\frac{1}{2}$. Diracov operátor je dôležitým objektom nekomutatívnej geometrie a tiež relativistickej kvantovej teórie poľa. Staviame na fakte, že *Poincarého algebra je pod-algebrou* $su\,(2,2)$. Potom môžeme použiť $su\,(2,2)$ na explicitnú konštrukciu reprezentácií Poincarého algebry. Ukazuje sa, že fundamentálna reprezentácia sama o sebe nestačí, pretože z nej nemožno vybudovať hmotnú teóriu. Toto má pôvod v tom, že *Casimirov element* $\hat{m}^2 = \hat{p}_\mu \hat{p}^\mu$ reprezentácie Poincarého algebry je rovný nule. Obdobná situácia nastáva pre duálnu reprezentáciu: $\tilde{m}^2 = \tilde{p}_\mu \tilde{p}^\mu = 0$. Vezmeme priamy súčet týchto dvoch reprezentácií a operátor hybnosti definujeme ako $P_\mu = \hat{p}_\mu + \tilde{p}_\mu$. Casimirov element M je teraz nenulový a interpretovaný ako hmotnosť. Toto nám umožňňuje zadefinovať relativistický Diracov operátor na fuzzy priestore. Hľadáme jeho vlstné funkcie a ukazuje sa, že netriviálna časť v porovnaní sa komutatívnym Diracovym operátorom spočíva v hľadaní vlstných funkcií operátora hybnosti P_μ.

Kľúčové slová: Fuzzy priestor, su(2,2), Diracov operátor, oscilátorová reprezentácia, relativistický

Abstract

We present approach of constructing relativistic Dirac operator of spin $\frac{1}{2}$ particle in fuzzy space, which is very important object for both non-commutative geometry and relativistic quantum field theory. We build on the fact that *Poincaré algebra is sub-algebra of* $su\,(2,2)$. We can thus use oscillator representation of $su\,(2,2)$ to explicitly construct representation of Poincaré algebra. During the process, we find out that just fundamental representation does not suffice, since it is impossible to define massive theory. This is due to the fact that *Casimir element* $\hat{m}^2 = \hat{p}_\mu \hat{p}^\mu$ of representation of Poincaré algebra is equal to zero. The same situation arises for dual representation: $\tilde{m}^2 = \tilde{p}_\mu \tilde{p}^\mu = 0$. We take direct sum of them, the momentum operator is then defined as $P_\mu = \hat{p}_\mu + \tilde{p}_\mu$. Casimir element M is now non-vanishing and interpreted as mass. This allows us to define relativistic Dirac operator in fuzzy space. Eigen-problem for the operator is then investigated. The non-trivial part in comparison with commutative Dirac operator is to find eigenfunctions of momentum operator P_μ.

Keywords: Fuzzy space, relativistic, Dirac operator, $su\,(2,2)$, oscillator representation

Contents

Introduction

It is well known fact that current well established theory of fundamental interactions, quantum field theory, is not UV finite and it fails to describe gravity since it is non-renormalizable. Lack of advances in qft approach to quantum gravity led to development of different approaches such as string theory and non-commutative geommetry - both of them often combined together.

Aspiring non-commutative theories often start by defining Dirac operator since one can learn a lot about underlying geometry by studying solutions of Dirac equation. This is due to the fact, that Alain Connes proved *spin manifold theorem* (see [5]) that given spectral triple and Charge conjugation operator, we can reconstruct whole manifold from spectrum of Dirac operator. We will pursue the path of constructing Dirac operator too, our aim is to define relativistic-covariant Dirac operator and investigate solutions to the free Dirac equation.

Our approach stands on the fact, that $su(2,2)$ algebra contains Poincaré algebra as sub-algebra. To obtain massive theory, we have to take direct sum of two $su(2,2)$ representations - fundamental and dual representation. This approach relies on oscillator representations of $su(2,2)$ (see [12] or [9]). We give detailed description of massless oscillator representation of $su(2,2)$, massive(constructed from two massless representations) as well as Poincaré algebra which it contains as sub-algebra. In our case, we defined 4-momentum operator P_μ which allows us to define free Dirac operator in fuzzy space.

Chapter 1 contains brief introduction into the problematics. Chapter 2 contains introduction to the $su(2,2)$ as well as explicit construction of infinite-dimensional representation of $su(2,2)$. In "physical" part of this work - chapters 4 and 3 we omitted most of the lengthy calculations and moved them to chapter 6 to highlight conceptual part and also to make reading little easier.

Chapter 1

Fuzzy spacess

1.1 Motivation

Reason to study non-commutative spaces is to make description of space fuzzy - that is to introduce some fundamental scale which determines the shortest observable length. This is usually done by making coordinates non-commutative, possible "positions" are then determined by their spectra. Non-commutativity necessarily leads to uncertainty relations between coordinates. This fact can be utilized e.g. to get regular description of quantum field theory. Another reason to introduce non-commutativity to the space is to naturally incorporate fundamental scale relevant for physics at Planck length. Very important theorem in non-commutative geometry is theorem of Alain Connes for (commutative) manifolds.

Theorem 1. *(*Alain Connes*) Given spectral triple*

1. $C^{\infty}(\mathcal{M})$ *smooth functions on manifold*
2. $L^{2}(\mathcal{M}, \mathcal{S})$ *spinor fileds on manifold*
3. D *Dirac operator*

together with charge conjugation and chirality operator, then metric g can be reconstructed from spectrum of the Dirac operator.

It is usually good practice to define non-commutative analogue of Dirac operator in the spirit of this theorem. We will, however, do this just very loosely.

To develop some intuition about non-commutative spaces, let us consider following two-dimensional non-commutative toy model. Define two non-commutative coordinates x_1,x_2 with commutation relation given by

$$[x_1, x_2] = \lambda \tag{1.1}$$

This coordinates can be constructed from annihilation and creation operators $x_1 = \sqrt{\lambda}a$, $x_2 = \sqrt{\lambda}a^\dagger$. Recall that for n-tuple of annihilation and creation operators $\left[a_i, a_j^\dagger\right] = \delta_{ij}$ holds. These operators act on auxiliary Hilbert space. Now we define non-commutative (analytic) function as:

$$\phi\left(x_1, x_2\right) = \sum_{m,n=0}^{\infty} c_{mn} x_1^n x_2^m \tag{1.2}$$

So far it does not seem much extra-ordinary, since it is obvious that the set of commutative analytic functions is of the same size as the set of non-commutative analytic functions. Let's look at product of the functions now and where commutativity takes place. We will use coherent states to get the job done. Recall that coherent state is eigenstate of annihilation operator i.e. $a\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle$. $\left|\alpha\right\rangle$ can be expressed as (see [15]):

$$\left|\alpha\right\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha)^n}{\sqrt{n!}} \left|n\right\rangle \tag{1.3}$$

but for our purposes here it is useful to redefine this as:

$$\left|\sqrt{\lambda}\alpha\right\rangle = e^{-\left|\alpha\sqrt{\lambda}\right|^2} \sum_{n=0}^{\infty} \frac{\left(\alpha\sqrt{\lambda}\right)^n}{\sqrt{n!}} \left|n\right\rangle \tag{1.4}$$

for which

$$x_1\left|\sqrt{\lambda}\alpha\right\rangle = \lambda\alpha\left|\sqrt{\lambda}\alpha\right\rangle \tag{1.5}$$

We now take symbol of non-commutative function $\phi\left(\bar{\chi}, \chi\right) = \left\langle\chi\right|\phi\left|\chi\right\rangle$ where $\chi = \alpha\lambda$ and we denote $\left|\sqrt{\lambda}\alpha\right\rangle$ as $\left|\chi\right\rangle$. Assignment of symbol is bijective. !!!!tuto nieco doplnit!!!! Let's take two non-commutative functions ζ and ψ. Their product is:

$$\zeta\psi = \sum_{m,n,i,j=0}^{\infty} c_{mn} k_{ij} x_1^m x_2^n x_1^i x_2^j \tag{1.6}$$

Using Wick's theorem for x_1 and x_2 and assigning symbol to the product of functions we get the following structure:

$$\left\langle\chi\right|\zeta\psi\left|\chi\right\rangle = \zeta\left(\bar{\chi}, \chi\right) + \lambda\left(\textit{single contractions}\right) + \lambda^2\left(\textit{double contractions}\right) + ... \tag{1.7}$$

We can see now, that as $\lambda \to 0$ we get commutative point-wise product.

1.2 Fuzzy spaces in our case

In this work we will not constrain ourselves to the specific non-commutative model, since our considerations are more general. Typically, non-commutativity (or fuzziness) is introduced as non-commutativity forced to coordinate functions (for example see [6]):

$$[x_k, x_l] = \Theta_{kl} \tag{1.8}$$

Where Θ_{kl} is not specified.

This in turn leads to non-commutativity of algebra of functions(since they are functions of coordinates) and uncertainty relations for them as well. Different commutation relations define, of course, different geometries. Common example of fuzzy space (with "fuzzy" Dirac operator) is fuzzy sphere or it's variations, see for example [8].

Often there is quantum theory constructed on fuzzy space to investigate some implications of non-commutativity. To get some picture of non-commutative space we shall construct space of functions. This is done by using two pairs of annihilation and creation operators a_i and a'^i (reasons for doing so will be discussed later). Recall that:

$$\begin{aligned} [a_i, a_j^\dagger] &= [a'^i, a_j'^\dagger] = \delta_{ij} \\ & [a_i, a_j'] = 0 \end{aligned} \tag{1.9}$$

Space of functions then consists of normal-ordered states of the form:

$$\Psi = \Psi\left(a, a^\dagger, a', a'^\dagger\right)$$

And more precisely:

$$\Psi = \sum C_{mnij} C'_{opkl} \left(a_1^\dagger\right)^m \left(a_2^\dagger\right)^n \left(a_1'^\dagger\right)^o \left(a_2'^\dagger\right)^p (a_1)^i (a_2)^j (a_1')^k (a_2')^l \tag{1.10}$$

Where a_α and a'_α are two pairs of annihilation (together with corresponding creation) operators. We additionally put one constraint for Ψ: it has to have the same number of annihilation and creation operators (of the same kind) - this effectively reduces two degrees of freedom from both C and C'.

Chapter 2

$spin\,(4,2)$, $su\,(2,2)$, oscillator representation

We shall begin this section by mentioning some important definitions and theorems. We give introduction to fundamental representations of $spin\,(4,2)$ and $su\,(2,2)$. In the end, we give oscillator representation of those mentioned algebras.

2.1 Basic definitions and theorems

Definition 1. ***An enveloping algebra*** $\mathfrak{A}$ *of lie algebra* $\mathfrak{g}$ *is an associative algebra in which can* $\mathfrak{g}$ *be embedded (denoting embedding as* Φ*) such that abstract brackets* $[f,g]$ *(*$f,\ g \in \mathfrak{g}$*) are realized as* $XY - YX$ *for* $X = \Phi(f)$ *and* $Y = \Phi(g)$*.*

Proposition 1. *The* ***Universal enveloping algebra*** $U(\mathfrak{g})$ *is the most general (maximal) enveloping algebra.*

This is rather vague statement, but we will give more precise definition soon. The key to this is to give precise meaning of "the most general". We will not, however, give complete theory behind universal enveloping algebra. For thorough description see [10] chapter 9.

Definition 2. *The* ***Universal enveloping algebra*** $U(\mathfrak{g})$ *is Tensor algebra over* $\mathfrak{g}$ *with condition* $A \otimes B - B \otimes A = [A,B]$ *imposed. That is quotient of Tensor algebra over* $\mathfrak{g}$ *with respect to ideal generated by elements of the form* $A \otimes B - B \otimes A - [A,B]$*.*

It is very intuitive that "the most general enveloping algebra" should be unique to be well defined. From Theorem 9.7. of [10] we can take this as a fact and in addition we can deduce that for any enveloping algebra $\mathfrak{A}$ there exist (unique) homomorphism $h: \mathfrak{A} \to U(\mathfrak{g})$ such that for $\phi : \mathfrak{g} \to \mathfrak{A}$ it holds that $h \circ \phi$ is the desired embedding. This gives rise to the fact, that we can compute Casimir operators (central operators for

$U(\mathfrak{g})$) in given representation entirely in terms of the basis matrices (of that algebra in given matrix representation), since embedding into algebra of matrices is embedding into associative algebra. Existence (and uniqueness) of mentioned homomorphism then gives clear relationship to $U(\mathfrak{g})$. This allows us to express *Casimir elements* in terms of some finite-dimensional associative algebra.

Theorem 2. ***Poincaré-Birkhoff-Witt (PBW)*** *Let $\mathfrak{g}$ be finite-dimensional Lie algebra with basis $e_1...e_n$ and $i : \mathfrak{g} \to U(\mathfrak{g})$ be injective homomorphism, and $X_j = i(e_j)$ then elements of the form:*

$$X_1^{n_1} \otimes ... \otimes X_k^{n_k}$$

are linearly independent and span $U(\mathfrak{g})$.

Definition 3. *The real* ***Clifford algebra*** *$Cl(p, q)$ is associative algebra with unit for which it additionally holds:*

$$e^i e^j = \eta^{ij} \tag{2.1}$$

where η^{ij} is bilinear form with signature (p, q). We will assume that η^{ij} is diagonal and non-degenerate.

Definition 4. ***spin*** $\boldsymbol{(m, n)}$ *is lie algebra of* ***Spin*** $\boldsymbol{(m, n)}$ *group that is positive-determinant invertible part of Clifford algebra.*

For us, $spin(4, 2)$ is relevant, since it is isomorphic to $su(2, 2)$.

2.2 $su(2, 2)$ algebra

2.2.1 Review of Poincaré algebra

Recall (see [18], we use, however, different signature convention) that for Poincaré group generators following commutation relations hold:

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{2.2}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k \tag{2.3}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k \tag{2.4}$$

$$[J_i, P_j] = i\epsilon_{ijk} P_k \tag{2.5}$$

$$[K_i, P_j] = -i\delta_{ij} P_0 \tag{2.6}$$

$$[J_i, P_0] = i[P_i, P_0] = 0 \tag{2.7}$$

$$[K_i, P_0] = -iP_i \tag{2.8}$$

where J_i, K_i are generators of rotations and boosts respectively. P_μ is generator of translations, four-vector and can be considered as momentum operator (given the right representation).

2.2.2 spin(4,2) and su(2,2)

Importance of $su\,(2,2)$ for this work lies in the fact, that it contains Poincaré algebra as an sub-algebra. $SU\,(2,2)$ group is real Lie group which preserves scalar product $\langle \psi^\dagger \phi \rangle$ with signature $(2,2)$ where ψ and ϕ are four-dimensional complex vectors. We can thus write element of $su\,(2,2)$ algebra as:

$$S = \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix} \tag{2.9}$$

where A, D are hermitian and B, $B^\dagger$ are general matrices. $su\,(2,2)$ algebra in fundamental representation is then given by following matrices:

$$S_{ij} = \frac{1}{2}\varepsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad S_{k4} = \frac{1}{2} \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix},$$
$$S_{0k} = \frac{i}{2} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad S_{45} = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$S_{k5} = \frac{1}{2} \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad S_{04} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$S_{05} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.10}$$

Where $i, j, k = 1..3$.
Now where can one see Poincaré group? Eagle-eyed reader should have already spotted that for example S_{ij} are generators of rotations. Boost matrices and generators of translations (momentum) are more intricate. One can straightforwardly (calculating commutators) verify that S_{0k} correspond to boosts and $S_{\mu 5} - S_{\mu 4}$ correspond to translation generators, let's call them p_μ. We have to stress out, that this is not the single option how to choose Poincaré subgroup, rather most obvious. From complexified version of the algebra we can get even richer structure.

Let's briefly discuss anti-fundamental (dual) representation of $su\,(2,2)$ denoted as S'_{ab}. For dual (matrix) representation of any Lie algebra holds that $\rho^\star(g) = -\rho^T(g)$ where g is an element of Lie algebra. It then follows that $S_{a5} = -S'_{a5}$. These representations are inequivalent. From anti-fundamental representation we choose Poincaré generators in the same manner as in fundamental representation and so $p'_\mu = S'_{\mu 5} - S'_{\mu 4}$. As it has already been mentioned, we will take $S \oplus S'$ for our considerations. This is, however, fundamental representation of $spin\,(4,2)$. Let us explicitly write generators of the dual representation.

$$S'_{ij} = -\frac{1}{2}\varepsilon_{ijk}\begin{pmatrix} \sigma_k^T & 0 \\ 0 & \sigma_k^T \end{pmatrix}, \qquad S'_{k4} = -\frac{1}{2}\begin{pmatrix} \sigma_k^T & 0 \\ 0 & -\sigma_k^T \end{pmatrix},$$
$$S'_{0k} = -\frac{i}{2}\begin{pmatrix} 0 & \sigma_k^T \\ \sigma_k^T & 0 \end{pmatrix}, \qquad S'_{45} = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$S'_{k5} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_k^T \\ -\sigma_k^T & 0 \end{pmatrix}, \qquad S'_{04} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$S'_{05} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.11}$$

And we can write down an element of fundamental representation of *spin* (4, 2) as:

$$\Sigma_{ab} = \begin{pmatrix} S_{ab} & 0 \\ 0 & S'_{ab} \end{pmatrix} \tag{2.12}$$

where S_{ab} and S'_{ab} denote fundamental and anti-fundamental representation of $su\,(2,2)$. This is very analogous to situation when one constructs spinor representation of Lorentz group. One can roughly say that $su\,(2,2)$ is to $so\,(4,2)$ (or $spin\,(4,2)$) what $sl\,(2,\mathbf{C})$ is to $so\,(3,1)$.

Now we construct oscillator representation of $su\,(2,2)$ (and also $spin\,(4,2)$ as consequence). Define operators on NC states $\hat{a}_i$, $\hat{a}_i^+$, $\hat{b}_i$, $\hat{a}_i^+$ for $i = 1,2$ as:

$$\begin{aligned} \hat{a}_i\Psi &= a_i\Psi \\ \hat{b}_i\Psi &= \Psi a_i \end{aligned} \tag{2.13}$$

and analogously for + operators. Commutation relations for these operators are:

$$\left[\hat{a}_i, \hat{a}_j^+\right] = -\left[\hat{b}_i, \hat{b}_j^+\right] = \delta_{ij} \tag{2.14}$$

Now we can arrange these operators into quadruple(s):

$$\hat{A}^T = (\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2)\,,\ \hat{A}^+ = (\hat{a}_1^+, \hat{a}_2^+, \hat{b}_1^+, \hat{b}_2^+) \tag{2.15}$$

And for oscillator representation we can write:

$$\hat{S}_{ab} \;=\; \hat{A}^+\,\Gamma\,S_{ab}\,\hat{A} \quad a,b = 0..5 \tag{2.16}$$

where $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is 4×4 matrix with 1 and -1 being 2×2 matrices.

This (or similar) approach can be seen for example in [12] or [9], we build mostly on the first one.

To construct oscillator realization S_{ab} of S'_{ab} we proceed very similarly. We now use second pair of a/c operators a' and $a'^{\dagger}$ introduced in (1.9). Operators $\hat{b}_i$ and $\hat{b}_i^+$ are defined for a'^{i} and $a_i'^{\dagger}$ in the same manner as operators in (2.13). There is one difference in operator realization for $\tilde{S}_{ab}$- Γ matrix has now swapped position to ensure that representation is dual:

$$\tilde{S}_{ab} = \tilde{A}^{+} S'_{ab} \Gamma \tilde{A} \quad a, b = 0..5 \tag{2.17}$$

where $\tilde{A}$ and $\tilde{A}^{+}$ are:

$$\tilde{A}^{T} = (\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2)\,, \;\; \tilde{A}^{+} = (\tilde{a}_1^{+}, \tilde{a}_2^{+}, \tilde{b}_1^{+}, \tilde{b}_2^{+}) \tag{2.18}$$

Theorem 3. *$\tilde{S}_{ab}$ and $\hat{S}_{ab}$ obey the same commutation relations as S_{ab} and S'_{ab} and thus they are indeed representations.*

Proof. We proceed for $\hat{S}$. Let S, K be from $su\,(2,2)$. Define $\bar{A} = A^{+}\Gamma$, then $\left[A_i, \bar{A}_i\right] = \delta_{ij}$ where $i, j = 1..4$. We write $\hat{S} = S_{ij}\bar{A}_i A_j$ and $\hat{K} = K_{mn}\bar{A}_m A_n$. Calculating commutator:

$$\begin{aligned}\left[\hat{S}, \hat{K}\right] =& S_{ij} K_{mn} \left\{\bar{A}_i A_j \bar{A}_m A_n - \bar{A}_m A_n \bar{A}_i A_j\right\} \\ =& S_{ij} K_{mn} \left\{\bar{A}_i \left(\delta_{mj} - \bar{A}_m\right) A_j A_n - \bar{A}_m \left(\delta_{in} - \bar{A}_i A_n\right) A_j\right\} \\ =& \bar{A}\,[S, K]\, A\end{aligned} \tag{2.19}$$

□

Theorem 4. *This representation is also unitary.*

Proof. Let us write the most general element of given lie algebra as:

$$\hat{S} = \left(u^{+}, -v^{+}\right) \begin{pmatrix} A & B \\ B^{\dagger} & D \end{pmatrix} (u_1, v_1)^{T} = u^{+}Au + u^{+}Bv - v^{+}B^{\dagger}u - v^{+}Dv.$$

Now computing hermitian conjugate of this operator we get

$$\hat{S}^{+} = -u^{+}Au - u^{+}Bv + v^{+}B^{\dagger}u + v^{+}Dv = -\hat{S}$$

$\hat{S}$ is thus anti-hermitian which is desired result. Since exponential of anti-hermitian operator gives unitary one. We now used mathematician's convention (this is the only time we use it). □

Remark: These considerations also make clear, why we defined NC wave functions with two pairs of annihilation and creation operators. The first pair a_i corresponds to fundamental representation and the second pair a'_i corresponds to anti-fundamental representation. This makes direct sum structure very visible.

2.3 Gauss decomposition

We now show technique how to boost non-commutative functions in operator representation.

2.3.1 Gauss decomposition for $sl(2,\mathbb{C})$

For $SL(2,\mathbb{C})$ we have only one constraint. Let g be general element of the group, then it is necessary for $det(g) = 1$ to hold. I the terms of Lie algebra $sl(2,\mathbb{C})$ it means that $Tr(X) = 0$ where X is from Lie algebra. We now choose (very common) basis for $sl(2,\mathbb{C})$.

$$H = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad K_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad K_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.20}$$

$$[H, K_\pm] = \pm K_\pm$$

Every element of $sl(2,\mathbb{C})$ (technically not every, however exception is set of zero measure) can be expressed as:

$$\begin{aligned} X &= \exp(tK_-)\exp(\tau H)\exp(sK_+) \\ &= \begin{pmatrix} e^{-\frac{1}{2}\tau} + ste^{\frac{1}{2}\tau} & te^{\frac{1}{2}\tau} \\ se^{\frac{1}{2}\tau} & e^{\frac{1}{2}\tau} \end{pmatrix} \\ &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma^{-1} & 0 \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \end{aligned} \tag{2.21}$$

From this we can see that for (almost, for reasons given above) general element of $sl(2,\mathbb{C})$ it is possible to write:

$$\begin{pmatrix} d & c \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ bd^{-1} & 1 \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}\begin{pmatrix} 1 & cd^{-1} \\ 0 & 1 \end{pmatrix} \tag{2.22}$$

We will use rather special case:

$$\begin{pmatrix} C & S \\ S & C \end{pmatrix} = \begin{pmatrix} 1 & SC^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C^{-1} & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} 1 & 0 \\ SC^{-1} & 1 \end{pmatrix} \tag{2.23}$$

We should remember that unit determinant condition is $C^2 - S^2 = 0$ so we choose natural parametrization $C = \cosh(t)$ and $S = \sinh(t)$. Additionally we introduce new variable $T = SC^{-1} = \tanh(T)$ and thus

$$\begin{pmatrix} C & S \\ S & C \end{pmatrix} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C^{-1} & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix} \tag{2.24}$$

2.3.2 Calculating boost

Since oscillator representation introduced in this work is complex, we are allowed to work also with complex linear combinations of generators introduced in (2.9) and thus to work with complexified algebra $su(2,2)_{\otimes\mathbb{C}}$. This is due to fact that complex representations are the same as representations of complexified algebra. We can then choose full $sl(2,\mathbb{C})$ from it. It is then given by matrices S_{45}, S_{04} and S_{05}.

We shall now examine how boost $\hat{B}_{45}$ acts on non-commutative functions.

$$\hat{B}_{45} = e^{-2iT\hat{S}_{45}} = e^{T\hat{K}_-}e^{2i\ln(C)\hat{S}_{05}}e^{T\hat{K}_+} \tag{2.25}$$

where $K_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $K_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. In oscillator representation then $\hat{K}_+ = \hat{a}^+\hat{b}$, $\hat{K}_- = \hat{b}^+\hat{a}$, $i\hat{S}_{05} = \frac{1}{2}i\left(\hat{a}^+\hat{a} + \hat{b}^+\hat{b}\right)$. We now introduce new labels $\hat{r} = -i\hat{S}_{05}$ and $\tau = \frac{1}{2}\ln(C)$. So we can rewrite above equation as:

$$\hat{B}_{45} = e^{-2iT\hat{S}_{45}} = e^{T\hat{K}_-}e^{\tau\hat{r}}e^{T\hat{K}_+} \tag{2.26}$$

We will now choose more convenient basis for non-commutative functions generated by $|n_1, n_2\rangle\langle m_1, m_2|$. Consequently we need to calculate matrix elements $\langle m_1, m_2|\hat{B}_{45}|n_1, n_2\rangle$. The calculation we are going to do is approached in somewhat simplified form, just for one pair of annihilation/creation operators, however without loss of generality, since calculation is done in exactly the same way with two pairs of operators. We calculate matrix elements for $\hat{K}_-$ acting on ψ.

$$\begin{aligned}
\langle n'| e^{T\hat{K}_-}\psi |n\rangle &= \sum_{k=0}^{\infty} \frac{T^k}{k!} \langle n'| a^k \psi a^{\dagger k} |n\rangle \\
&= \sum_{k=0}^{\infty} T^k \sqrt{\frac{(k+n')!\,(k+n)!}{k!n'!k!n!}} \langle n'+k| \psi |n+k\rangle \\
&= \sum_{k=0}^{\infty} T^k \sqrt{\binom{k+n'}{k}\binom{k+n}{k}} \langle n'+k| \psi |n+l\rangle
\end{aligned} \tag{2.27}$$

And in the similar fashion for $\hat{K}_+$:

$$\begin{aligned}
\langle n'| e^{T\hat{K}_+}\psi |n\rangle &= \sum_{k=0}^{\infty} \frac{T^k}{k!} \langle n'| a^{\dagger k} \psi a^k |n\rangle \\
&= \sum_{k=0}^{\infty} T^k \sqrt{\frac{n'!n!}{k!\,(n-k)!k!\,(n'-k)!}} \langle n'-k| \psi |n-k\rangle \\
&= \sum_{k=0}^{\infty} T^k \sqrt{\binom{k+n'}{k}\binom{k+n}{k}} \langle n'+k| \psi |n+l\rangle
\end{aligned} \tag{2.28}$$

Let's investigate action of $\hat{K}_\pm$ on non-commutative functions now.

$$\begin{aligned}\Phi^T_{K_+} = e^{T\hat{K}_+}\Phi &= \sum_{k=0}^{\infty}\frac{T^k}{k!}\left(\hat{a}^+_\alpha\hat{b}_\alpha\right)^k\Phi \\ &= \sum_{k=0}^{\infty}\frac{T^k}{k!}\sum_{l=0}^{k}\binom{k}{l}\left(\hat{a}^+_1\hat{b}_1\right)^l\left(\hat{a}^+_2\hat{b}_2\right)^{k-l}\Phi \\ &= \sum_{k=0}^{\infty}T^k\sum_{k_1+k_2=k}\frac{a_1^{\dagger k_1}a_2^{\dagger k_2}}{\sqrt{k_1!k_2!}}\Phi\frac{a_1^{k_1}a_2^{k_2}}{\sqrt{k_1!k_2!}}\end{aligned} \tag{2.29}$$

and practically the same calculation goes for $\hat{K}_-$:

$$\begin{aligned}\Phi^T_{K_-} = e^{T\hat{K}_-}\Phi &= \sum_{k=0}^{\infty}\frac{T^k}{k!}\left(\hat{a}_\alpha\hat{b}^+_\alpha\right)^k\Phi \\ &= \sum_{k=0}^{\infty}T^k\sum_{k_1+k_2=k}\frac{a_1^{k_1}a_2^{k_2}}{\sqrt{k_1!k_2!}}\Phi\frac{a_1^{\dagger k_1}a_2^{\dagger k_2}}{\sqrt{k_1!k_2!}}\end{aligned} \tag{2.30}$$

Due to the fact that for Φ we have a constraint regarding equal number of annihilation and creation operators of the same kind, it is easy to see, that Φ acts diagonally on auxiliary Fock space i.e. $\Phi\left|n_1n_2\right\rangle = \Phi_{n_1n_2}\left|n_1n_2\right\rangle$. Since (2.26) preserves this property, it is natural to investingate the action in terms of basis $\left|n_1n_2\right\rangle\left\langle n_1n_2\right|$. Using similar technique as in (2.27) and (2.29) we derive how constituents of (2.26) act on Ψ.

$$\begin{aligned}\Phi^T_{K_+}\left|n_1,n_2\right\rangle &= \sum_{k=0}^{\infty}T^k\sum_{k_1+k_2=k}\binom{n_1}{k_1}\binom{n_2}{k_2}\Phi_{n_1-k_1,n_2-k_2}\left|n_1,n_2\right\rangle \\ \Phi^T_{K_-}\left|n_1,n_2\right\rangle &= \sum_{k=0}^{\infty}T^k\sum_{k_1+k_2=k}\binom{k_1+n_1}{k_1}\binom{k_2+n_2}{k_2}\Phi_{n_1+k_1,n_2+k_2}\left|n_1,n_2\right\rangle \\ e^{-\tau\hat{r}}\Phi\left|n_1,n_2\right\rangle &= e^{-\tau(n_1+n_2+1)}\Phi_{n_1,n_2}\left|n_1,n_2\right\rangle\end{aligned} \tag{2.31}$$

Putting everything together, we present grand formula for boosted solution:

$$\begin{aligned}\hat{B}_{45}\Phi = &\sum_{n_1,n_2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k_1+k_2=k}\sum_{i_1+i_2=j}\binom{k_1+n_1}{k_1}\binom{k_2+n_2}{k_2}\times \\ &\times\binom{k_1+n_1}{i_1}\binom{k_2+n_2}{i_2}T^{k+j}\Phi_{n_1+k_1-i_1,n_2+k_2-i_2}\left|n_1,n_2\right\rangle\left\langle n_1,n_2\right|\end{aligned} \tag{2.32}$$

To get the function which will eventually correspond to non-zero three-momentum, we need to rotate this solution. We choose rotation given by $\hat{R}_i = \begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}$ Oscillator representation for this operator then gives:

$$\hat{R} = \hat{a}^+\sigma_i\hat{a} - \hat{b}^+\sigma_i\hat{b} \tag{2.33}$$

and rotation, using BCH formula:

$$e^{it\hat{R}}\Psi = e^{itR_i}\Psi e^{-itR_i} \tag{2.34}$$

where $R_i = a^\dagger \sigma a$ and a_i are two annihilation operators on the auxiliary Fock space. We stress out that for dual representation approach is the same, it is merely interchange of some symbols and labels.

Chapter 3

Momentum operator

3.1 Definition

Since we need oscillator representation of $spin\,(4,2)$ on NC wave functions (or fields, if one deals with the field theory) we can define two "small" momentum operators $\hat{p}_\mu$ and $\tilde{p}_\mu$ from translation generators of fundamental and anti-fundamental representations of $su\,(2,2)$:

$$\begin{aligned} \hat{p}_\mu &= \hat{S}_{\mu 5} - \hat{S}_{\mu 4} \\ \tilde{p}_\mu &= \tilde{S}_{\mu 5} - \tilde{S}_{\mu 4} \end{aligned} \tag{3.1}$$

Let's look on the operator somewhat closer, first we take generator of translations:

$$p_0 = \frac{1}{2}\left(S_{05} - S_{04}\right) = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \tag{3.2}$$

for time translations and

$$p_k = \frac{1}{2}\left(S_{k5} - S_{k4}\right) = \frac{1}{4}\begin{pmatrix} -\sigma_i & \sigma_i \\ -\sigma_i & \sigma_i \end{pmatrix} \tag{3.3}$$

for space translations.
Now for operator representation we get for $\hat{p}_0$:

$$\begin{aligned} \hat{p}_0 = &\quad \left(\hat{a}_1^+, \hat{a}_2^+, -\hat{b}_1^+, -\hat{b}_2^+\right)\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\left(\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2\right)^T \\ = &\quad \tfrac{1}{4}\left\{\hat{a}_\alpha^+\hat{a}_\alpha + \hat{b}_\alpha^+\hat{b}_\alpha + \hat{a}_\alpha^+\hat{b}_\alpha + \hat{b}_\alpha^+\hat{a}_\alpha\right\} \end{aligned} \tag{3.4}$$

and for spatial part:

$$\hat{p}_i = \frac{1}{4}\sigma^i_{\alpha\beta}\left\{\hat{a}_\alpha^+\hat{a}_\beta + \hat{b}_\alpha^+\hat{b}_\beta + \hat{a}_\alpha^+\hat{b}_\beta + \hat{b}_\alpha^+\hat{a}_\beta\right\} \tag{3.5}$$

Splitting into components momentum operator acts on non-commutative functions ϕ as:

$$\begin{aligned}
\hat{p}_0\phi =& \frac{1}{4}\{[a_1^+,[a_1,\phi]] + [a_2^+,[a_2,\phi]]\} \\
\hat{p}_1\phi =& -\frac{1}{4}\{[a_2^+,[a_1,\phi]] + [a_1^+,[a_2,\phi]]\} \\
\hat{p}_2\phi =& -\frac{1}{4}i\{[a_2^+,[a_1,\phi]] - [a_1^+,[a_2,\phi]]\} \\
\hat{p}_3\phi =& -\frac{1}{4}\{[a_1^+,[a_1,\phi]] - [a_2^+,[a_2,\phi]]\}
\end{aligned} \tag{3.6}$$

Similarly for $\tilde{S}$ representation we get:

$$\begin{aligned}
\tilde{p}_0 =& \left(\tilde{a}_1^+,\tilde{a}_2^+,\tilde{b}_1^+,\tilde{b}_2^+\right)\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\left(\tilde{a}_1,\tilde{a}_2,-\tilde{b}_1,-\tilde{b}_2\right)^T \\
=& -\tfrac{1}{4}\left\{\tilde{a}_\alpha^+\tilde{a}_\alpha + \tilde{b}_\alpha^+\tilde{b}_\alpha + \tilde{a}_\alpha^+\tilde{b}_\alpha + \tilde{b}_\alpha^+\tilde{a}_\alpha\right\}
\end{aligned} \tag{3.7}$$

and for spatial part:

$$\begin{aligned}
\tilde{p}_{1,3} =& -\frac{1}{4}\sigma_{\alpha\beta}^{1,3}\left\{\tilde{a}_\alpha^+\tilde{a}_\beta + \tilde{b}_\alpha^+\tilde{b}_\beta + \tilde{a}_\alpha^+\tilde{b}_\beta + \tilde{b}_\alpha^+\tilde{a}_\beta\right\} \\
\tilde{p}_2 =& \frac{1}{4}\sigma_{\alpha\beta}^{2}\left\{\tilde{a}_\alpha^+\tilde{a}_\beta + \tilde{b}_\alpha^+\tilde{b}_\beta + \tilde{a}_\alpha^+\tilde{b}_\beta + \tilde{b}_\alpha^+\tilde{a}_\beta\right\}
\end{aligned} \tag{3.8}$$

And again, splitting into components results in:

$$\begin{aligned}
\tilde{p}_0\phi' =& -\frac{1}{4}\{[a_1'^+,[a_1',\phi']] + [a_2'^+,[a_2',\phi']]\} \\
\tilde{p}_1\phi' =& \frac{1}{4}\{[a_2'^+,[a_1',\phi']] + [a_1'^+,[a_2',\phi']]\} \\
\tilde{p}_2\phi' =& -\frac{1}{4}i\{[a_2'^+,[a_1',\phi']] - [a_1'^+,[a_2',\phi']]\} \\
\tilde{p}_3\phi' =& \frac{1}{4}\{[a_1'^+,[a_1',\phi']] - [a_2'^+,[a_2',\phi']]\}
\end{aligned} \tag{3.9}$$

Where ϕ' does not mean derivative, it denotes that function corresponds to dual representation. It can be proven (see appendix) that:

$$\hat{p}^\mu\hat{p}_\mu = \tilde{p}^\mu\tilde{p}_\mu = 0 \tag{3.10}$$

Once again analogy with Lorentz group shows up - it is impossible to define massive theory with just fundamental or anti-fundamental representation alone. In the case of Lorentz group we have massless left-handed or right-handed spinors, in our case of $spin\,(4,2)$ we have "left-handed" or "right-handed" twistors. Twistor is 4-dimensional analogue of spinor.

In the end, we use $\hat{p}_\mu$ and $\tilde{p}_\mu$ to construct momentum operator of the massive theory:

$$P_\mu = \hat{p}_\mu + \tilde{p}_\mu \tag{3.11}$$

Since P_μ has been constructed from generators of translations it is also generator of translations. From the fact, that p_μ and p'_μ are also 4-vectors, P_μ is 4-vector too. Let's define operator of mass $M = P^\mu P_\mu$. Simplifying $(\hat{p}_\mu + \tilde{p}_\mu)(\hat{p}^\mu + \tilde{p}^\mu)$ we get:

$$M^2 = 2\hat{p}_\mu \tilde{p}^\mu \tag{3.12}$$

and thus M has, of course, the same eigenfunctions as P_μ.

3.2 Eigenvalue problem for the momentum operator

In addition to solve eigen-problem for spinors as in commutative case, we have to find eigenfunctions of momentum operator (momentum in NC Dirac equation is operator on NC states) if we want to find eigenfunctions of the Dirac operator. We need to do that just for $\hat{p}_0$ and $\tilde{p}_0$. This is justified by the fact, that we can consider states with zero three-momentum and then boost them in the right direction to obtain solution for any three-momentum. It is, of course, impossible to have zero three-momentum (for nonzero four-momentum) just for "small momentum operators", since they correspond to massless fields. For P^μ it is possible and we will indeed capitalize this property.

Since NC states posses direct sum structure and in addition $[a_1, a_2] = [a'_1, a'_2] = 0$ holds, together with assumption of normal ordering of NC states and equal number of annihilation and creation operators, we can introduce possibilities for ansatz:

$$\Phi = \begin{cases} \phi_1(N_1)\tilde{\phi}_1(N'_1) \\ \phi_2(N_2)\tilde{\phi}_2(N'_2) \\ \phi_1(N_1)\tilde{\phi}_2(N'_2) \\ \phi_2(N_2)\tilde{\phi}_1(N'_1) \end{cases} \tag{3.13}$$

where $N_1 = a_1^\dagger a_1$, $N_2 = a_2^\dagger a_2$ and analogously for N'_i. ϕ_i and $\tilde{\phi}_i$ are normally ordered. Now we approach just for ϕ_1. This is without loss of generality, considering that task is the same for ϕ_2 and $\tilde{\phi}_i$ - it is merely just interchange of symbols. $\hat{p}_0$ acts on ϕ_1 as:

$$\hat{p}_0\Psi = -\frac{1}{4}\left[a_1^+, [a_1, \phi_1]\right] \tag{3.14}$$

And we want to solve:

$$\hat{p}_0\Psi = E\Psi \tag{3.15}$$

Considering the fact that commutators act as derivatives (by "erasing" a/c operators) this equation can be rewritten into differential form:

$$x\chi''(x) + \chi'(x) + 4E\chi(x) = 0 \tag{3.16}$$

This equation has the solution:

$$\chi(x) = C_1 J_0\left(4\sqrt{Ex}\right) + C_2 Y_0\left(4\sqrt{Ex}\right) \tag{3.17}$$

where J_0 and Y_0 are Bessel functions of the first and second kind. The full operator solution is then normal ordered operator-evaluated χ i.e. $\phi_\alpha(N_\alpha) = \ :\chi(N_\alpha): $. For N_1 then holds(see appendix):

$$\begin{aligned} \phi_1(N_1) : \chi(N_1) : = \sum_{l=0}^{\infty} \frac{(-16E)^l}{2^{2l}(l!)^2} a_1^{\dagger l} a_1^l + \\ + \frac{2}{\pi}\left\{\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{\infty}\binom{k}{i}\frac{(-1)^{2k+j-i-1}4^{2j+i}E^{l+i}}{k2^{2j}(j!)^2}\left(a_1^\dagger\right)^{i+j}a_1^{i+j}\right\} + \\ + \frac{2}{\pi}\left\{\sum_{k=1}^{\infty}(-1)^{k-1}H_k\frac{4^k E}{(k!)^2}\left(a_1^+\right)^k a_1^k\right\} \end{aligned} \tag{3.18}$$

What does this solution correspond to? Take a look at how operators $\hat{p}_i$ act on $\phi_1(N_1)$. Since $[a_\alpha, a_\beta] = 0$ and $\left[a_\alpha, a_\beta^\dagger\right] = \delta_{\alpha\beta}$ it directly follows that (considering 3.14):

$$\begin{aligned} \hat{p}_1\phi_1(N_1) =& 0 \\ \hat{p}_2\phi_1(N_1) =& 0 \\ \hat{p}_3\phi_1(N_1) =& -E\phi_1(N_1) \end{aligned} \tag{3.19}$$

and for $\phi_2(N_2)$ we have (again, see 3.14):

$$\begin{aligned} \hat{p}_1\phi_2(N_2) =& 0 \\ \hat{p}_2\phi_2(N_2) =& 0 \\ \hat{p}_3\phi_2(N_2) =& E\phi_1(N_1) \end{aligned} \tag{3.20}$$

There is obvious interpretation for this - one gets solution for particle moving along *z axis* together with solution for particle "moving" in opposite direction. The same behavior we get for $\tilde{p}_\mu$:

$$\begin{aligned} \tilde{p}_1\tilde{\phi}_1(N_1) =& 0 \\ \tilde{p}_2\tilde{\phi}_1(N_1) =& 0 \\ \tilde{p}_3\tilde{\phi}_1(N_1) =& -E\tilde{\phi}_1(N_1) \end{aligned} \tag{3.21}$$

and for $\tilde{\phi}_2(N_2)$ the similar relations hold

$$\begin{aligned} \tilde{p}_1\tilde{\phi}_2(N_2) =& 0 \\ \tilde{p}_2\tilde{\phi}_2(N_2) =& 0 \\ \tilde{p}_3\tilde{\phi}_2(N_2) =& E\tilde{\phi}_1(N_1) \end{aligned} \tag{3.22}$$

Finally, for possible solution(s) we write:

$$\Psi\left(N_1, N_2, N_1', N_2'\right) = \begin{cases} \phi_1\left(N_1\right)\tilde{\phi}\left(N_1'\right) \\ \phi_2\left(N_2\right)\tilde{\phi}_2\left(N_2'\right) \\ \phi_1\left(N_1\right)\tilde{\phi}_2\left(N_2'\right) \\ \phi_2\left(N_2\right)\tilde{\phi}_1\left(N_1'\right) \end{cases} \tag{3.23}$$

Look at eigenfunctions of P_μ now. They are, of course, eigenfunctions of $\hat{p}_\mu$ and $\tilde{p}_\mu$. We are especially interested for solutions of motionless particle. Those are $\Psi_1 = \phi_2\left(N_2\right)\tilde{\phi}_1\left(N_1'\right)$ and $\Psi_2 = \phi_1\left(N_1\right)\tilde{\phi}_2\left(N_2'\right)$. Then P_μ acts as:

$$\begin{aligned} P_0\Psi_1\left(N_1, N_2, N_1', N_2'\right) =& 2E\Psi_1\left(N_1, N_2, N_1', N_2'\right) \\ P_0\Psi_2\left(N_1, N_2, N_1', N_2'\right) =& 2E\Psi_2\left(N_1, N_2, N_1', N_2'\right) \\ P_3\Psi_1\left(N_1, N_2, N_1', N_2'\right) =& 0 \\ P_3\Psi_2\left(N_1, N_2, N_1', N_2'\right) =& 0 \end{aligned} \tag{3.24}$$

and it, of course, also vanishes on for P_1 and P_2. This is solution with zero three momentum. To obtain full solution, formula derived in the last section chapter 2 should be used to boost and rotate solution.

Chapter 4

Noncommutative Dirac operator

The work on non-commutative Dirac operator has already been done. The nontrivial part, comparing to commutative Dirac operator, is eigen-problem for momentum operator P_μ. We will briefly discuss Dirac equation for the sake of completeness.

4.1 Dirac operator and Dirac equation

Recall that Dirac operator in Minkowski space is $\gamma_\mu \partial^\mu$ and then the Dirac equation:

$$(i\gamma_\mu \partial^\mu - m)\psi(x) = 0 \tag{4.1}$$

and fourier-transforming equation we get:

$$(\gamma_\mu p^\mu - m)\psi(p) = 0 \tag{4.2}$$

this is the form that non-commutative version shall take too (with p_μ, however, being an operator). Dirac equation is typically solved in a following way:

1. take fourier transform of Dirac equation (4.2)
2. take ansatz $u(p)^T = (u_1, u_2)$ and solve for u_1, u_2
3. write solution in symmetric form $u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\zeta \\ \sqrt{p \cdot \bar{\sigma}}\zeta \end{pmatrix}$

where ζ is two-componnent spinor.

4.2 Non-commutative version

We introduce Dirac equation right in p-representation in the similar way as in commutative theory - we have no choice but ordinary gamma matrices since we want spin $\frac{1}{2}$ Lorentz invariant equation.

$$(\gamma^\mu P_\mu - M)\Psi_p = 0 \tag{4.3}$$

Taking $M\Psi_p = m\Psi_p$ and $P_\mu\Psi_p = p_\mu\Psi_p$ we see it takes form of ordinnary Dirac equation. The major difference here is that components of bi-spinor Ψ_p are noncommutative functions, solutions from the (3.23) or their boosted/rotated versions (see 2.34 and 2.32).

Chapter 5

Conclusions

Logic of our work relies on the fact, that $su\,(2,2)$ contains Poincaré algebra as subalgebra. We proceeded straightforwardly in the beginning with fundamental representation of $su\,(2,2)$. We took translation generators $S_{\mu 4} - S_{\mu 5}$ to define momentum operator. This is done by oscillator representation on non-commutative functions. The representation acts on space of NC functions, constructed from two pairs of annihilation and creation operators. First pair for fundamental and second pair for anti-fundamental (dual) representation. Lesson gathered from representation theory of $so\,(3,1)$ (mass mixes left-handed with right-handed components of spinors) can also be seen here. Momentum operator $\hat{p}_\mu$ has vanishing square $\hat{p}_\mu \hat{p}^\mu$ and we need to take direct sum with dual representation. The momentum operator P_μ is then sum of operators from fundamental and dual representation $\hat{p}_\mu$ and $\tilde{p}_\mu$. In this representation we possess Casimir element of Poincaré algebra constructed in terms of operators $\hat{p}_\mu$ and $\tilde{p}_\mu$ this is naturally interpreted as mass. In chapters 2, 3 and 4 we investigated (and mostly solved) eigen-problem for Dirac operator. The spinor part is the same as in commutative case, however non-commutativity of functions introduces some difficulties. We overcame them and developed method how to deal with the problem, at least in P-representation.

Chapter 6

Appendix: calculations and formulas

We now calculate the result (3.10). For the sake of clarity we define two-dimensional complex vectors $\hat{a} = (\hat{a}_1, \hat{a}_2)^T$ and $\hat{b} = \left(\hat{b}_1, \hat{b}_2\right)^T$. Now momentum operator can be written as follows:

$$\begin{aligned}
\hat{p}_0 =& \frac{1}{2}\left(\hat{a}^+, -\hat{b}^+\right)\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\left(\hat{a}, \hat{b}\right)^T\\
=& \frac{1}{2}\left(\hat{a}^+ - \hat{b}^+\right)(a-b)\\
\hat{p}_i =& \frac{1}{2}\left(\hat{a}^+, -\hat{b}^+\right)\begin{pmatrix}-\sigma_i & \sigma_i\\ -\sigma_i & \sigma_i\end{pmatrix}\left(\hat{a}, \hat{b}\right)^T\\
=& -\frac{1}{2}\left(\hat{a}^+ - \hat{b}^+\right)\sigma_i\left(\hat{a} - \hat{b}\right)
\end{aligned} \tag{6.1}$$

Lorentz square of $\hat{p}_\mu$ is:

$$\hat{p}_\mu \hat{p}^\mu = \hat{p}_0\hat{p}_0 - \hat{p}_1\hat{p}_1 - \hat{p}_2\hat{p}_2 - \hat{p}_3\hat{p}_3 \tag{6.2}$$

Since σ_3 is diagonal matrix it is natural to group $\hat{p}_0$ (contains identity matrix) with $\hat{p}_3$ (which contains σ_3) and also to group $\hat{p}_0$ and $\hat{p}_0$ since they contain anti-diagonal matrices σ_1 and σ_2.

$$\hat{p}_\mu \hat{p}^\mu = (\hat{p}_0\hat{p}_0 - \hat{p}_3\hat{p}_3) - (\hat{p}_2\hat{p}_2 + \hat{p}_1\hat{p}_1) \tag{6.3}$$

Now it is natural to rewrite the equation using basic product formulas:

$$\hat{p}_\mu \hat{p}^\mu = (\hat{p}_0 - \hat{p}_3)(\hat{p}_0 + \hat{p}_3) - (\hat{p}_1 - i\hat{p}_2)(\hat{p}_1 + i\hat{p}_2) \tag{6.4}$$

We shall now calculate first and second term separately. For the first one:

$$\begin{aligned}
\hat{p}_0 \pm \hat{p}_3 =& \frac{1}{2}\left(\hat{a}^+ - \hat{b}^+\right)(1 \mp \sigma_3)\left(\hat{a} - \hat{b}\right)\\
=& \frac{1}{2}\left(\hat{a}^+ - \hat{b}^+\right)\begin{pmatrix}1 \mp 1 & 0\\ 0 & 1 \pm 1\end{pmatrix}\sigma_3\left(\hat{a} - \hat{b}\right)\\
=& \begin{cases}\left(\hat{a}_2^+ - \hat{b}_2^+\right)\left(\hat{a}_2 - \hat{b}_2\right) & case\ +\\ \left(\hat{a}_1^+ - \hat{b}_1^+\right)\left(\hat{a}_1 - \hat{b}_1\right) & case\ -\end{cases}
\end{aligned} \tag{6.5}$$

and for the second one:

$$\begin{aligned}\hat{p}_1 \pm i\hat{p}_2 =& \frac{1}{2}\left(\hat{a}^+ - \hat{b}^+\right)(1 \mp \sigma_3)(a-b) \\ =& \frac{1}{2}\left(\hat{a}^+ - \hat{b}^+\right)\begin{pmatrix} 0 & 1\pm 1 \\ 1\mp 1 & 0\end{pmatrix}\sigma_3 (a-b) \\ =& \begin{cases} -\left(\hat{a}_1^+ - \hat{b}_1^+\right)\left(\hat{a}_2 - \hat{b}_2\right) & case\ + \\ -\left(\hat{a}_2^+ - \hat{b}_2^+\right)\left(\hat{a}_1 - \hat{b}_1\right) & case\ - \end{cases}\end{aligned} \tag{6.6}$$

And thus we finally arrive at desired conclusion

$$\begin{aligned}\hat{p}_\mu \hat{p}^\mu =& \left(\hat{a}_2^+ - \hat{b}_2^+\right)\left(\hat{a}_2 - \hat{b}_2\right)\left(\hat{a}_1^+ - \hat{b}_1^+\right)\left(\hat{a}_1 - \hat{b}_1\right) \\ &- \left(\hat{a}_2^+ - \hat{b}_2^+\right)\left(\hat{a}_1 - \hat{b}_1\right)\left(\hat{a}_1^+ - \hat{b}_1^+\right)\left(\hat{a}_2 - \hat{b}_2\right) = 0\end{aligned} \tag{6.7}$$

Where we used that $\left(\hat{a}_1 - \hat{b}_1\right)$ commutes with $\left(\hat{a}_1^+ - \hat{b}_1^+\right)$. To prove this, we use commutation relations for a and b $\left[\hat{a}_i, \hat{a}_j^+\right] = \delta_{ij}$ and $\left[\hat{b}_i, \hat{b}_j^+\right] = -\delta_{ij}$. Simple calculation then shows that:

$$\begin{aligned}\left(\hat{a}_1^+ - \hat{b}_1^+\right)\left(\hat{a}_1 - \hat{b}_1\right) =& \hat{a}_1^+\hat{a}_1 - \hat{b}_1^+\hat{a}_1 - \hat{a}_1^+\hat{b}_1 + \hat{b}_1^+\hat{b}_1 \\ =& \hat{a}_1\hat{a}_1^+ - 1 - \hat{b}_1^+\hat{a}_1 - \hat{a}_1^+\hat{b}_1 + \hat{b}_1\hat{b}_1^+ + 1 \\ =& \left(\hat{a}_1 - \hat{b}_1\right)\left(\hat{a}_1^+ - \hat{b}_1^+\right)\end{aligned} \tag{6.8}$$

This proof, of course, passes for general product $\left(\hat{a}_i^+ - \hat{b}_i^+\right)\left(\hat{a}_i - \hat{b}_i\right)$. We now repeat previous procedure for $\tilde{p}_\mu$ with some minor modifications. Recall that $p' = S'_{\mu 5} - S'_{\mu 4}$ and in oscillator representation $\tilde{p}_0 = A^+ p'_\mu \Gamma A$. For momentum operator then holds:

$$\begin{aligned}\tilde{p}_0 =& \frac{1}{2}\left(\tilde{a}^+, \tilde{b}^+\right)\begin{pmatrix} -1 & -1 \\ 1 & 1\end{pmatrix}\left(\tilde{a}, -\tilde{b}\right) \\ =& -\frac{1}{2}\left(\tilde{a}^+ - \tilde{b}^+\right)\left(\tilde{a} - \tilde{b}\right)\end{aligned} \tag{6.9}$$

$$\begin{aligned}\tilde{p}_1 =& \frac{1}{2}\left(\tilde{a}^+, \tilde{b}^+\right)\begin{pmatrix} \sigma_1 & \sigma_1 \\ -\sigma_1 & -\sigma_1\end{pmatrix}\left(\tilde{a}, -\tilde{b}\right) \\ =& \frac{1}{2}\left(\tilde{a}^+ - \tilde{b}^+\right)\sigma_1\left(\tilde{a} - \tilde{b}\right)\end{aligned} \tag{6.10}$$

$$\begin{aligned}\tilde{p}_2 =& \frac{1}{2}\left(\tilde{a}^+, \tilde{b}^+\right)\begin{pmatrix} -\sigma_2 & -\sigma_2 \\ \sigma_2 & \sigma_2\end{pmatrix}\left(\tilde{a}, -\tilde{b}\right) \\ =& -\frac{1}{2}\left(\tilde{a}^+ - \tilde{b}^+\right)\sigma_2\left(\tilde{a} - \tilde{b}\right)\end{aligned} \tag{6.11}$$

$$\begin{aligned}\tilde{p}_3 =& \frac{1}{2}\left(\tilde{a}^+, \tilde{b}^+\right)\begin{pmatrix} \sigma_3 & \sigma_3 \\ -\sigma_3 & -\sigma_3\end{pmatrix}\left(\tilde{a}, -\tilde{b}\right) \\ =& \frac{1}{2}\left(\tilde{a}^+ - \tilde{b}^+\right)\sigma_3\left(\tilde{a} - \tilde{b}\right)\end{aligned} \tag{6.12}$$

And to calculate Lorentz square $\tilde{p}_\mu \tilde{p}^\mu$ we proceed similarly as in previous case.

$$\tilde{p}_\mu \tilde{p}^\mu = (\tilde{p}_0 \tilde{p}_0 - \tilde{p}_3 \tilde{p}_3) - (\tilde{p}_2 \tilde{p}_2 + \tilde{p}_1 \tilde{p}_1) \tag{6.13}$$

Now it is natural to rewrite the equation using basic product formulas:

$$\tilde{p}_\mu \tilde{p}^\mu = (\tilde{p}_0 - \tilde{p}_3)(\tilde{p}_0 + \tilde{p}_3) - (\tilde{p}_1 - i\tilde{p}_2)(\tilde{p}_1 + i\tilde{p}_2) \tag{6.14}$$

We, again, calculate first and second term separately. For the first one:

$$\begin{aligned}
\tilde{p}_0 \pm \tilde{p}_3 =& -\frac{1}{2}\left(\tilde{a}^+ - \tilde{b}^+\right)(1 \mp \sigma_3)\left(\tilde{a} - \tilde{b}\right) \\
=& \frac{1}{2}\left(\tilde{a}^+ - \tilde{b}^+\right)\begin{pmatrix} 1 \mp 1 & 0 \\ 0 & 1 \pm 1 \end{pmatrix}\sigma_3\left(\tilde{a} - \tilde{b}\right) \\
=& \begin{cases} -\left(\tilde{a}_2^+ - \tilde{b}_2^+\right)\left(\tilde{a}_2 - \tilde{b}_2\right) & case\ + \\ -\left(\tilde{a}_1^+ - \tilde{b}_1^+\right)\left(\tilde{a}_1 - \tilde{b}_1\right) & case\ - \end{cases}
\end{aligned} \tag{6.15}$$

and for the second one:

$$\begin{aligned}
\tilde{p}_1 \pm i\tilde{p}_2 =& \frac{1}{2}\left(\tilde{a}^+ - \tilde{b}^+\right)(1 \mp \sigma_3)\left(\tilde{a} - \tilde{b}\right) \\
=& \frac{1}{2}\left(\tilde{a}^+ - \tilde{b}^+\right)\begin{pmatrix} 0 & 1 \mp 1 \\ 1 \pm 1 & 0 \end{pmatrix}\left(\tilde{a} - \tilde{b}\right) \\
=& \begin{cases} \left(\tilde{a}_1^+ - \tilde{b}_1^+\right)\left(\tilde{a}_2 - \tilde{b}_2\right) & case\ + \\ \left(\tilde{a}_2^+ - \tilde{b}_2^+\right)\left(\tilde{a}_1 - \tilde{b}_1\right) & case\ - \end{cases}
\end{aligned} \tag{6.16}$$

Let's derive differential equation introduced in (3.16). We first calculate the commutator $[a, \phi]$ where ϕ is normal ordered function of N_1.

$$\begin{aligned}
\left[a_1^\dagger, \left(a_1^\dagger\right)^m (a_1)^n\right] =& a_1^\dagger \left(a_1^\dagger\right)^m (a_1)^n - \left(a_1^\dagger\right)^m (a_1)^n a_1^\dagger \\
=& \left(a_1^\dagger\right)^{m+1} (a_1)^n - \left(a_1^\dagger\right)^m (a_1)^{n-1}\left(1 + a_1^\dagger a_1\right) = ... \\
=& -n\left(a_1^\dagger\right)^m (a_1)^{n-1}
\end{aligned} \tag{6.17}$$

where ellipsis denotes repetitive use of commutation relation $\left[a_i, a_j^\dagger\right] = \delta_{ij}$ until one arrives to the final formula. Using the same approach we calculate:

$$\left[a_1, \left(a_1^\dagger\right)^m (a_1)^n\right] = m\left(a_1^\dagger\right)^{m-1} (a_1)^n \tag{6.18}$$

This means that commutator acts as derivative (which is not at all surprising since commutators obey Leibniz rule). Let us now ask question how this commutator acts on normal ordered functions of number operator N_1. This is done by setting $m = n$ in previous equations (6.17 and 6.18). One can now easily see that:

$$\begin{aligned} \left[a_1, :N_1^n:\right] = n : N_1^{n-1} : a_1 \implies : \left[a_1, \chi(N_1)\right] : =: \chi'(N_1)\, a_1 : \\ \left[: N_1^n :, a_1^\dagger\right] = n a_1^\dagger : N_1^{n-1} : \implies : \left[\chi(N_1), a_1^\dagger\right] : =: a_1^\dagger \chi'(N_1) : \end{aligned} \tag{6.19}$$

Now consider expression $\phi(N_1)\, a_1$ where normal ordered function ϕ is χ'. By using Leibniz rule we have:

$$\left[a_1^\dagger, \phi(N_1)\, a_1\right] = \left[a_1^\dagger, \phi(N_1)\right] a_1 + \phi(N_1)\left[a^\dagger, a_1\right] \tag{6.20}$$

and after normal ordering:

$$\begin{aligned} : \left[a_1^\dagger, \phi(N_1)\, a_1\right] : &= \quad - : a_1^\dagger \phi' a_1 : -\phi \\ &= -: N_1 \chi''(N_1) - \chi\prime(N_1) : \end{aligned} \tag{6.21}$$

Recall that:

$$\begin{aligned} \hat{p}_0 &= \tfrac{1}{4}\left\{\hat{a}_\alpha^+ \hat{a}_\alpha + \hat{b}_\alpha^+ \hat{b}_\alpha + \hat{a}_\alpha^+ \hat{b}_\alpha + \hat{b}_\alpha^+ \hat{a}_\alpha\right\} \\ \hat{p}_i &= \tfrac{1}{4}\sigma_{\alpha\beta}^i \left\{\hat{a}_\alpha^+ \hat{a}_\beta + \hat{b}_\alpha^+ \hat{b}_\beta + \hat{a}_\alpha^+ \hat{b}_\beta + \hat{b}_\alpha^+ \hat{a}_\beta\right\} \end{aligned}$$

From this, one can finally write equivalent differential equation for $\hat{p}_0\phi_1 = E\phi_1$ as

$$\begin{aligned} \hat{p}_0\phi_1 = & -\tfrac{1}{4}\{: N_1 \phi_1''(N_1) + \phi_1'(N_1) :\} = E\phi_1(N_1) \\ & \implies \zeta\chi''(\zeta) + \chi(\zeta) + 4E\chi(\zeta) = 0 \end{aligned} \tag{6.22}$$

And operator solution $\phi_1(N_1)$ is then obtained from normal ordered operator evaluated solution $\chi(\zeta)$. Finally $\phi_1(N_1) =: \chi(N_1) :$.

To prove result in (3.18) we first write identities for Bessel functions(exhausting information can be found in [1]):

$$\begin{aligned} J_0(z) &= \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l}(l!)^2} z^{2l} \\ Y_0(z) &= \frac{2}{\pi}\left\{\ln\left(\frac{1}{2}z\right)_0 (z) + \sum_1^{\infty} (-1)^{k+1} H_k \frac{4^{-k} z^{2k}}{(k!)^2}\right\} \end{aligned} \tag{6.23}$$

where $H_k = \sum_{i=1}^k \frac{1}{i}$ are harmonic numbers. The first term in expression for Y_0 does not look good since it contains logarithm. This can be circumvented by exponentiating argument of logarithm by 2 and multiplicating by factor $\frac{1}{2}$. As consequence, we do not have to deal with square root of operator. Writing logarithm as power series

$$\ln(X) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}(x-1)^k}{k} \tag{6.24}$$

and using binomial theorem

$$(X-1)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} X^i. \tag{6.25}$$

Inserting $4\sqrt{EN_1}$ combining everything together we get the desired result.

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