

Abstract

Cubic graphs without bridges can be classified by their *perfect matching index*—the minimum number of perfect matchings required to cover its edges—denoted by $\pi(G)$. Snarks—bridgeless cubic graphs with $\pi(G) \geq 4$ —are further classified into those with $\pi(G) = 4$ and those with $\pi(G) \geq 5$. In this thesis, we call cubic graphs with $\pi(G) \leq 4$ *coverable* and those with $\pi(G) \geq 5$ *uncoverable*. This thesis investigates uncoverability of snarks by four perfect matchings.

First, we explore the connection between covering as many edges as possible with four perfect matchings and covering every edge with four (not necessarily perfect) matchings. We show, both theoretically and algorithmically, that, for every bridgeless cubic graph G , the minimum total number of vertices left uncovered across the four matchings in a matching cover of G is at most twice the minimum number of edges left uncovered by four perfect matchings of G . We construct an infinite family of nontrivial (i.e., cyclically 4-edge-connected and of girth at least 5) uncoverable snarks for which these two values coincide and can be arbitrarily large. In addition, we show examples of such snarks where the values differ. Furthermore, we show that every bridgeless cubic graph has a cover by four matchings containing at least two perfect matchings, strengthening a classical result of Petersen, which guarantees only one perfect matching in such a cover. Additionally, we relate the existence of such a cover containing three perfect matchings to a famous conjecture of Fan and Raspaud.

Next, we investigate the existence of *extended tetrahedral flows* on bridgeless cubic graphs, defined as a nowhere-zero $\mathbb{Z}_2^4$ -flow whose values are restricted to elements of weight 1 or 2, where the weight of an element of $\mathbb{Z}_2^4$ is the number of its zero coordinates. We relate the existence of such flows to the 5-Cycle Double Cover Conjecture and a famous conjecture of Fan and Raspaud.

Finally, we explore covering a cubic graph G by odd factors (spanning subgraphs in which every vertex has degree 1 or 3). We investigate the smallest number of degree-3 vertices across four odd factors that constitute a cover of G . We establish relationships between this parameter and the minimum number of defective lines in extended tetrahedral flows of G .

Keywords: snark, perfect matching, nowhere-zero flow, cyclic connectivity, edge coloring