GROUP ACTIONS AND HOMOGENEOUS SPACES

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Foreword

This booklet presents the proceedings of an international topology symposium that was held at the Faculty of Mathematics, Physics, and Informatics of Comenius University in Bratislava, September 7-11, 2009. The aim of this conference – the Bratislava Topology Symposium "Group Actions and Homogeneous Spaces" – was to discuss, mainly, finite group actions on manifolds, torus actions on manifolds, and topology of homogeneous spaces. There were several plenary lectures (60 minutes long), each surveying a research area chosen by the corresponding speaker within the framework of one of the three main topics and outlining prospective ideas or directions of study as well. In addition, there were shorter talks (30 minutes long) presenting research results of the speakers. Participants with the corresponding titles of talks given at the Symposium are listed below; the plenary speakers were Bogusław Hajduk, Matthias Kreck, Tibor Macko, Mikiya Masuda, Taras Panov, Peter Teichner, and Aleksy Tralle. Further details can be found at http://thales.doa.fmph.uniba.sk/bts/ or http://sites.google.com/site/conferencepage/bts2009/.

These Proceedings contain papers which were presented at the conference and some related papers. Two further papers, which were prepared in relation to this Symposium, will appear in the journal Mathematica Slovaca. All papers have been refereed, and we take this opportunity to express our sincere thanks to the referees. We thank Peter Zvengrowski for technical editing of the contributions.

Thanks also go to the Mathematical Institute of the Slovak Academy of Sciences for help with arranging the accommodation for several participants.

Finally, we should like to thank everyone who contributed to the success of the Symposium.

Bratislava / Okayama / Poznań, October 2010

Július Korbaš, Masaharu Morimoto, and Krzysztof Pawałowski
Organizing Committee

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- Sören Illman (University of Helsinki, Helsinki, Finland), *Equivariant Alexander-Spanier cohomology for non-Lie group actions*
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- Július Korbaš (Comenius University, Bratislava, Slovakia), *A new bound for the cup-length of zero-cobordant manifolds*
- Matthias Kreck (Hausdorff Research Institute for Mathematics, Bonn University, Bonn, Germany), *Simply connected asymmetric manifolds*
- Shintaro Kuroki (KAIST, Daejeon, South Korea), *On projective bundles over small covers*

- Tibor Macko (Mathematical Institute, University of Münster, Münster, Germany, and Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia), *Surgery classification of lens spaces*

- Wacław Marzantowicz (Adam Mickiewicz University, Poznań, Poland), *Homotopical theory of periodic points of periodic homeomorphisms on closed surfaces*

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CONTENTS

A note on the characteristic rank of a smooth manifold

Ľudovít Balko and Július Korbaš

ABSTRACT. This paper presents some results, using the characteristic rank recently introduced by the second named author, on those smooth manifolds which can serve as total spaces of smooth fibre bundles with fibres totally non-homologous to zero with respect to $\mathbb{Z}_2$. As the main results, first, some upper and lower bounds for the characteristic rank of those total spaces which need not be null-cobordant are derived; then, bounds for the characteristic rank of null-cobordant total spaces are deduced. Examples are shown, where the upper and lower bounds coincide; thus these bounds cannot be improved in general. All examples of manifolds considered are homogeneous spaces.

1 Introduction

Our aim in this note is to present some results on those smooth manifolds which can serve as total spaces of smooth fibre bundles. More precisely, we mainly shall deal with some situations, where a new homotopy invariant of smooth closed manifolds called the characteristic rank, introduced by the second named author in [4], brings an interesting piece of information. For this, we concentrate on smooth fibre bundles with fibres totally non-homologous to zero: given a smooth fibre bundle $p : E \longrightarrow B$ with total space E, base space B and fibre F we recall (see, e.g., [7, p. 124]) that F is said

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Keywords and phrases: smooth manifold, characteristic rank, Stiefel-Whitney characteristic class, fibre (fiber) bundle, fibre (fiber) totally non-homologous to zero, null-cobordant (zero-cobordant) manifold.

[2]Part of this research was carried out while Ľ. Balko was a member of a team supported in part by the grant agency VEGA and J. Korbaš was a member of two research teams supported in part by the grant agency VEGA and a member of a bilateral Slovak-Slovenian $0005-08$ team supported in part by the grant agency APVV.

to be totally non-homologous to zero (in E) with respect to a given coefficient ring R if the fibre inclusion $i : F \to E$ induces an epimorphism, $i^* : H^*(E;R) \to H^*(F;R)$, in cohomology.

In the sequel, we shall always understand $R = \mathbb{Z}_2$ and write just $H^i(X)$ instead of $H^i(X;\mathbb{Z}_2)$ for the ith $\mathbb{Z}_2$-cohomology group of X. In addition, all manifolds, thus also the total spaces, fibres and base spaces of smooth fibre bundles, will be (supposed to be, even if we do not mention it explicitly) smooth, connected and closed; all examples of manifolds considered will be homogeneous spaces. Note that for our purposes we may take any fibre $F_b = \{x \in E; p(x) = b\}$, where $b \in B$, in the role of F mentioned above, since B is path connected.

One may think of various types of conditions which must be satisfied by the $\mathbb{Z}_2$-cohomology of the total space E of any smooth fibre bundle $p : E \to B$ with base B and fibre F, if F should be totally non-homologous to zero in E. Among the well-known ones are (see [7], about the Leray-Hirsch theorem), for instance, that the $\mathbb{Z}_2$-Poincaré polynomial $P(E;t) = \sum_i \dim_{\mathbb{Z}_2} H^i(E)t^i$ must be the product $P(B;t)P(F;t)$, that the induced homomorphism $p^* : H^*(B) \to H^*(E)$ must be a monomorphism or that the kernel of $i^* : H^*(E) \to H^*(F)$, where $i : F \to E$ is the fibre inclusion, must be the ideal generated by $p^*(H^+(B))$, where $H^+(B) = \sum_{i>0} H^i(B)$.

For specific manifolds in the role of F or B, we may be able to derive specific conditions. To give a not so well known (as compared to, for instance, spheres or projective spaces) example: for the real Grassmann manifold $G_{2^s+4,4} \cong O(2^s+4)/(O(4)\times O(2^s))$ ($s \geq 3$), consisting of 4-dimensional vector subspaces in $\mathbb{R}^{2^s+4}$, one calculates ([1]) that the height of the third Stiefel-Whitney class $w_3 \in H^3(G_{2^s+4,4})$ of the canonical 4-plane bundle over $G_{2^s+4,4}$ is equal to 2^s+1; in other words, we have $w_3^{2^s+1} \neq 0$, but $w_3^{2^s+2} = 0$. Therefore if E were the total space of a smooth fibre bundle over B with $G_{2^s+4,4}$ as fibre F, this fibre being totally non-homologous to zero, then there must exist an element $x \in H^3(E)$ such that $i^*(x) = w_3$, $x^{2^s+1} \neq 0$, and x^{2^s+2} must lie in the ideal generated by $p^*(H^+(B))$.

As is known (see, for instance, [6]), the Stiefel-Whitney characteristic classes $w_i(M) \in H^i(M)$ of a (smooth, closed, connected) manifold M are identified with the Stiefel-Whitney classes of its tangent bundle TM, thus $w_i(M) = w_i(TM)$. These characteristic classes are crucial in studying several fundamental properties of M. Most notably, M is orientable if and only if $w_1(M) = 0$. But, as already indicated in [4], it turns out that the degree up to which the cohomology algebra $H^*(M)$ is generated by the Stiefel-Whitney classes of M also carries useful information.

More precisely, in [4] the second named author defined the characteristic rank, briefly $\mathrm{charrank}(M)$, of a d-dimensional manifold M, to be the largest integer k, $0 \leq k \leq d$, such that each element of the cohomology group $H^j(M)$ with $j \leq k$ can be expressed as a polynomial in the Stiefel-Whitney classes of M. For instance, if M is orientable and $H^1(M) \neq 0$, then we have $\mathrm{charrank}(M) = 0$. For more results on the values of characteristic rank, see [4].

The usefulness of the characteristic rank is already clear from the following theorem. By $\mathrm{cup}(M)$ we denote the $\mathbb{Z}_2$-cup-length of the manifold M, hence the maximum of all numbers c such that there exist, in positive degrees, cohomology classes $a_1, \ldots, a_c \in H^*(M)$ such that their cup product $a_1 \cup \cdots \cup a_c$ is nonzero. In addition, let r_M denote the smallest number such that the reduced cohomology group $\tilde{H}^{r_M}(M)$ does not vanish (we note that $0 < r_M \leq d$ since M is a connected d-dimensional manifold).

Theorem 1.1. (Korbaš [4, Theorem 1.1]) *Let M be a closed, smooth, connected, d-dimensional, unorientedly null-cobordant manifold. Then we have that*

$$\mathrm{cup}(M) \leq 1 + \frac{d - \mathrm{charrank}(M) - 1}{r_M}. \tag{1}$$

In the following section, we shall show that a new type of numerical condition which is satisfied by the total space E, if F is totally non-homologous to zero in E, can be obtained by using the characteristic rank. As the main results, we shall first derive (in Theorem 2.1) some upper and lower bounds for the characteristic rank of those total spaces which need not be null-cobordant (zero-cobordant), and then we shall deduce (in Theorem 2.2) bounds for the characteristic rank of null-cobordant total spaces. In addition, (infinitely many) non-trivial fibre bundles will be exhibited for which our upper and lower bounds coincide; thus these bounds cannot be improved in general.

2 The characteristic rank for smooth fibre bundles with fibre totally non-homologous to zero

2.1 General total spaces

In this subsection, as the first of our main results, we give some bounds for the characteristic rank of the total space of any smooth fibre bundle with fibre totally non-homologous to zero; the total space need not be null-cobordant.

Theorem 2.1. *Let $p : E \to B$ be a smooth fibre bundle with fibre F totally non-homologous to zero. Then we have that*

$$\min\{r_B, r_F\} - 1 = r_E - 1 \leq \mathrm{charrank}(E) \leq \mathrm{charrank}(F).$$

Proof. It is clear that $\mathrm{charrank}(E) \geq r_E - 1$. Since for the Poincaré polynomials we now have $P(E;t) = (1 + t^{r_E} + \dots) = P(B;t)P(F;t) = (1 + t^{r_B} + \dots)(1 + t^{r_F} + \dots)$, we see that $r_E = \min\{r_B, r_F\}$, and so it is true that $\mathrm{charrank}(E) \geq r_E - 1 = \min\{r_B, r_F\} - 1$.

It remains to prove that $\mathrm{charrank}(E) \leq \mathrm{charrank}(F)$. Take any cohomology class $x \in H^k(F)$ with $k \leq \mathrm{charrank}(E)$. Since $i^* : H^*(E) \to H^*(F)$ is an epimorphism, there exists some $y \in H^k(E)$ such that $i^*(y) = x$. Thanks to the fact that $k \leq \mathrm{charrank}(E)$, we have that $y = Q(w_1(E), w_2(E), \dots)$ for some polynomial Q.

For any smooth fibre bundle we have $TE \cong p^*(TB) \oplus \kappa$, where κ is the vector bundle along the fibres (so that $i^*(\kappa) \cong TF$). As a consequence, for the Stiefel-Whitney characteristic classes we have $i^*(w_t(E)) = w_t(F)$ for all t, thus implying that

$$x = i^*(y) = i^*(Q(w_1(E), w_2(E), \dots)) = Q(w_1(F), w_2(F), \dots).$$

This finishes the proof. □

Remark 2.1. For any (smooth, closed, connected) manifolds M and N, we have two obvious trivial fibre bundles with the same total space $M \times N$. As a special case of the preceding theorem, we obtain that

$$\min\{r_M, r_N\} - 1 \leq \mathrm{charrank}(M \times N)$$

and

$$\mathrm{charrank}(M \times N) \leq \min\{\mathrm{charrank}(M), \mathrm{charrank}(N)\}.$$

Remark 2.2. As a consequence of the fact that $i^*(w_t(E)) = w_t(F)$ for all t, any smooth fibre bundle $p : E \to B$ with fibre F such that $\mathrm{charrank}(F) = \dim(F)$ has fibre totally non-homologous to zero; see [3] for further details. Hence Theorem 2.1 applies, in particular, to all fibre bundles such that $\mathrm{charrank}(F) = \dim(F)$.

Remark 2.3. The following example shows one of possible uses of Theorem 2.1 and testifies that, in general, the bounds for $\mathrm{charrank}(E)$ given by Theorem 2.1 cannot be improved.

Example 2.1. We calculate the characteristic rank for the complex flag manifolds $F(1, 1, n-2) \cong U(n)/(U(1) \times U(1) \times U(n-2))$. We recall that $F(1, 1, n-2)$ may be interpreted to consist of triples (S_1, S_2, S_3), where S_i are mutually orthogonal vector subspaces in $\mathbb{C}^n$ such that $\dim_{\mathbb{C}}(S_1) = \dim_{\mathbb{C}}(S_2) = 1$ and $\dim_{\mathbb{C}}(S_3) = n-2$. Then one has a smooth fibre bundle over the complex Grassmann manifold $\mathbb{C}G_{n,2} \cong U(n)/(U(2) \times U(n-2))$ (consisting of complex 2-dimensional vector subspaces in $\mathbb{C}^n$), $p : F(1, 1, n-2) \to \mathbb{C}G_{n,2}$, $p(S_1, S_2, S_3) = S_1 \oplus S_2$. One can see in several ways (for instance, by applying [7, Ch. 3, Lemma 4.5]) that the fibre, the complex projective space $\mathbb{C}P^1$ (known to be diffeomorphic to the 2-dimensional sphere S^2), is totally non-homologous to zero with respect to $\mathbb{Z}_2$. Of course, we have $\mathrm{charrank}(\mathbb{C}P^1) = 1$. Now for $E = F(1, 1, n-2)$, $B = \mathbb{C}G_{n,2}$, $F = \mathbb{C}P^1$, we have $r_B = r_F = 2$, hence the lower and upper bounds given by Theorem 2.1 coincide, and we obtain that $\mathrm{charrank}(F(1, 1, n-2)) = 1$.

2.2 Null-cobordant total spaces

For any null-cobordant manifold E, one has (cf. [4]) $\mathrm{charrank}(E) < \dim(E)$. For the characteristic rank of such a manifold, if it serves as the total space of a smooth fibre bundle with fibre totally non-homologous to zero, we now derive, as the second of our main results, the following.

Theorem 2.2. *Let $p : E \to B$ be a smooth fibre bundle with E null-cobordant and with fibre F totally non-homologous to zero. Then we have that*

$$\min\{r_B, r_F\} - 1 = r_E - 1 \leq \mathrm{charrank}(E) \leq \min\{u_{B,F}, \mathrm{charrank}(F)\},$$

where $u_{B,F} = \dim(B) + \dim(F) - 1 - \min\{r_B, r_F\}(\mathrm{cup}(B) + \mathrm{cup}(F) - 1)$.

Proof. We have now by Horanská and Korbaš [2, Lemma, p. 25] that

$$\mathrm{cup}(E) \geq \mathrm{cup}(B) + \mathrm{cup}(F).$$

At the same time, by Theorem 1.1 we have the inequality (1) for $M = E$. Thus we obtain the inequality

$$r_E \mathrm{cup}(B) + r_E \mathrm{cup}(F) \leq r_E + \dim(E) - \mathrm{charrank}(E) - 1,$$

and this gives the upper bound for charrank(E) stated in the theorem, if we take into account that $r_E = \min\{r_B, r_F\}$, $\dim(E) = \dim(B) + \dim(F)$ and that, in addition, by Theorem 2.1 we have the inequality charrank$(E) \leq$ charrank(F). The lower bound is the same as in Theorem 2.1. This finishes the proof. □

The following example is a non-trivial application of Theorem 2.2 and also gives evidence that, in general, the bounds for charrank(E) given by Theorem 2.2 are sharp.

Example 2.2. We again calculate, this time in a different way (as compared to Example 2.1), the characteristic rank for the complex flag manifolds $F(1, 1, n-2)$. We now take a smooth fibre bundle over the complex projective space $\mathbb{C}P^{n-1}$, $p : F(1, 1, n-2) \to \mathbb{C}P^{n-1}$, $p(S_1, S_2, S_3) = S_1$. Its fibre, the complex projective space $\mathbb{C}P^{n-2}$, is totally non-homologous to zero with respect to $\mathbb{Z}_2$ (this can be seen in several ways; for instance, apply [7, Ch. 3, Lemma 4.5]). There is an obvious smooth fixed point free involution on $F(1, 1, n-2)$, interchanging S_1 and S_2 for every $(S_1, S_2, S_3) \in F(1, 1, n-2)$; in other words, the group $\mathbb{Z}_2$ acts smoothly and without fixed points on $F(1, 1, n-2)$. As a consequence (cf. [5]), the flag manifold $F(1, 1, n-2)$ is null-cobordant. We have $\mathrm{cup}(\mathbb{C}P^k) = k$ (see for instance [8, Theorem 15.33]). Now for $E = F(1, 1, n-2)$, $B = \mathbb{C}P^{n-1}$, $F = \mathbb{C}P^{n-2}$, we have $r_B = r_F = 2$ and $u_{B,F} = 2n-2+2n-4-1-2(n-1+n-2-1) = 1$, hence the lower and upper bounds given by Theorem 2.2 coincide, and we obtain that charrank$(F(1, 1, n-2)) = 1$.

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The Borsuk-Ulam theorem for manifolds, with applications to dimensions two and three

Daciberg L. Gonçalves, Claude Hayat, and Peter Zvengrowski

ABSTRACT. In this work a generalization of the Borsuk-Ulam theorem, for maps of a finite-dimensional CW-complex X with a fixed point free cellular involution τ into euclidean space $\mathbb{R}^n$, is considered. The case in which X is a manifold of dimension m is studied, in particular for $m = 2$ and $m = 3$. Examples of the various possibilities that can occur in dimension $m = 3$ are illustrated with Seifert manifolds.

1 Introduction

The theorem now known as the Borsuk-Ulam theorem seems to have first appeared in a paper of Lyusternik and Schnirel'man [22] in 1930, then in a paper of Borsuk [4] in 1933 (in a footnote to his paper Borsuk mentions that the theorem was posed as a conjecture by S. Ulam). One of the most familiar statements (Borsuk's Satz II) is that for any continuous map $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$. Letting $\tau_a : S^n \to S^n$ be the antipodal map, the Borsuk-Ulam theorem can thus be thought of as a coincidence theorem for the maps f and $f\tau_a$.

The theorem has many equivalent forms and generalizations, one obvious generalization being to replace S^n and its antipodal involution τ_a by any finite dimensional CW-complex X which admits a fixed point free cellular involution τ, and ask whether

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$f(x) = f(\tau(x))$ must hold for some $x \in X$. The original theorem and its generalizations have many applications in topology and other branches of mathematics. For example, since the pioneering work of Lovász [21] and Bárány [1] in 1978, there are important applications to combinatorics and graph theory. An excellent general reference for these applications as well as the theorem itself is the book of Matoušek [23]. In the area of analysis the "ham sandwich theorem" (cf. [23]) is a well known application related to measure theory, while the 1978 paper [5] of Browne and Binding gives an application to multiparameter eigenvalue problems.

We will state many equivalent forms of this generalized Borsuk-Ulam theorem in Section 2 (Proposition 2.2). The validity of the theorem in this generality is more delicate since it may now depend on the involution (assuming such exists). For a simple example of this take the disjoint union $X = S^1 \sqcup S^1$ and consider continuous maps $X \to \mathbb{R}^1$. If τ is the involution that is the antipodal map on each S^1, then the Borsuk-Ulam theorem clearly holds, whereas if one takes τ to be the involution that interchanges the two S^1's then it fails (e.g. let f take one S^1 into $\{0\}$ and the other into $\{1\}$). Less trivial examples where the result may depend on the involution are given in [11], where the case X a closed surface was studied. Using Lemma 2.4 the generalized Borsuk-Ulam theorem takes the following form : for a connected m-dimensional CW-complex X with fixed point free cellular involution τ, find the largest n such that any continuous map $f : X \to \mathbb{R}^n$ has a coincidence $f(x) = f\tau(x)$ for some $x \in X$. Furthermore, Lemma 2.4 implies that this largest n satisfies $1 \leq n \leq m$. In [12] the case where X is a 3-space form was considered, and in the present work we undertake the study for all closed connected m-manifolds with involution, in particular for $m = 2, 3$ a complete solution is obtained.

In Section 3 the results from Section 2 are applied to the case where $n = 2$ and to the case $n = m = \dim(X)$. Necessary and sufficient conditions for $n = 2$ and $n = m = \dim(X)$ will be given in Theorems 3.1 and 3.4, respectively, where in the case $n = \dim(X)$ we also assume that X is a closed connected n-dimensional manifold.

The two dimensional case is discussed in Section 4. For three dimensions, examples such as the classical Borsuk-Ulam triple $(S^3, \tau_a, 3)$ are easily treated by the method of this work. In Section 5, several other examples illustrating all the possibilities for $X = M^3$, a 3-manifold, will be given, using Seifert manifolds. These examples also illustrate the scope of the general programme of describing the Borsuk-Ulam theorem for Seifert manifolds, for which this note can be considered an initial step.

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2 Preliminaries

In this section we shall first discuss the Borsuk-Ulam theorem for a pair (X, τ), where X is a connected m-dimensional CW-complex, and τ a free cellular involution of X (meaning $\tau^2 = \mathrm{id}_X$, τ is fixed point free and permutes the cells of X). The space X is taken to be connected to avoid trivial examples where the Borsuk-Ulam theorem fails, such as the example in the Introduction. Also note that $\mathrm{card}(X) \geq 2$ must necessarily hold for such an involution to exist. By "map" we shall always mean "continuous map".

Definition 2.1. With (X, τ) as above, we call (X, τ, n) a Borsuk-Ulam triple if for any map $f : X \to \mathbb{R}^n$ there is at least one point $x \in X$ such that $f(x) = f(\tau(x))$ (i.e. x is a point of coincidence of f and $f\tau$).

We shall also (equivalently) say that *the Borsuk-Ulam theorem holds for the triple* (X, τ, n).

Before stating the next proposition we recall and set up notations for a few standard concepts from homotopy theory and the theory of fibre bundles.

We call (X, ν) a $\mathbb{Z}_2$-space if $\mathbb{Z}_2$ acts on X via a self homeomorphism ν. A $\mathbb{Z}_2$-map $f : X \to Y$ between two $\mathbb{Z}_2$-spaces is simply a $\mathbb{Z}_2$-equivariant map, written $f : X \xrightarrow{\mathbb{Z}_2} Y$.

Following [23], p. 95, the $\mathbb{Z}_2$-index of (X, ν) is defined as

$$\mathrm{ind}_{\mathbb{Z}_2}(X, \nu) := \min\{n \in \{0, 1, 2, ..., \infty\} \mid \text{there exists } f : X \xrightarrow{\mathbb{Z}_2} S^n\}.$$

If an involution is not free then the $\mathbb{Z}_2$-index is always ∞. When one considers fixed point free involutions $\mathrm{ind}_{\mathbb{Z}_2}(X)$ will be finite for finite dimensional CW-complexes X, as can be seen from Lemma 2.4(3) and Proposition 2.2 (1) $\equiv$ (3) below.

In this article (X,τ) is a $\mathbb{Z}_2$-space, τ is a fixed point free cellular involution and the $\mathbb{Z}_2$-maps are maps $f : X \to \mathbb{R}^n$ (or $f : X \to S^{n-1}$) with $f\tau(x) = -f(x)$ for all $x \in X$. We shall always write $W := X/\tau$ for the base space of the two-fold covering projection induced by τ (in the classical case $X = S^n$, τ is the antipodal map, and $W = \mathbb{R}P^n$). Note that $\tau : X \twoheadrightarrow W$ is a principal $\mathbb{Z}_2$-bundle which is not trivial since X is connected. Since τ is cellular, W is also an m-dimensional CW-complex.

For any principal $\mathbb{Z}_2$-bundle $X \twoheadrightarrow W$, we write $\gamma : W \to \mathbb{R}P^\infty = B\mathbb{Z}_2$ for the classifying map of the bundle. It is unique up to homotopy and can be taken as induced from a $\mathbb{Z}_2$-map $\Gamma : X \xrightarrow{\mathbb{Z}_2} S^\infty$. Denoting the generator of $H^1(\mathbb{R}P^\infty;\mathbb{Z}_2) \approx \mathbb{Z}_2$ by y, the characteristic class of the principal bundle is then $x = \gamma^*(y) \in H^1(W;\mathbb{Z}_2)$. Since the bundle is non-trivial, as remarked in the previous paragraph, it follows that $x \neq 0$ and that $\gamma_\# : \pi_1(W) \to \pi_1\mathbb{R}P^\infty = \mathbb{Z}_2$ is surjective.

Having established these definitions and notations, we now begin with a result stating several equivalent formulations of Definition 2.1, followed by comments about the proof of the various equivalences. By equivariant we shall always mean $\mathbb{Z}_2$-equivariant.

Proposition 2.2. *Let (X,τ) be as above. Then the following are equivalent.*

(1) *The Borsuk-Ulam theorem holds for the triple (X,τ,n).*

(2) *For every equivariant map $f : X \to \mathbb{R}^n$, $0 \in \mathrm{Im}(f)$.*

(3) *There is no equivariant map $f : X \to S^{n-1}$.*

(4) *There is no map $f : W \to \mathbb{R}P^{n-1}$ such that the pull-back of the non-trivial class $y \in H^1(\mathbb{R}P^{n-1};\mathbb{Z}_2)$ is the characteristic class x of the $\mathbb{Z}_2$-bundle $X \twoheadrightarrow W$.*

(5) *The classifying map $\gamma : W \to \mathbb{R}P^\infty$ does not compress to $\mathbb{R}P^{n-1}$.*

(6) *One has $\mathrm{ind}_{\mathbb{Z}_2}(X,\tau) \geq n$,*

(7) *If $\{U_1,\ldots,U_{n+1}\}$ is any covering of X by $n+1$ open sets, then for some $x \in X$ and some j, U_j contains the pair $\{x,\tau(x)\}$.*

(8) *If $\{A_1,\ldots,A_{n+1}\}$ is any covering of X by $n+1$ closed sets, then for some $x \in X$ and some j, A_j contains the pair $\{x,\tau(x)\}$.*

(9) *If* $\{B_1, \ldots, B_{n+1}\}$ *is any covering of* X *by* $n+1$ *sets, each of which is a closed set or an open set, then for some* $x \in X$ *and some* j, B_j *contains the pair* $\{x, \tau(x)\}$.

(10) *There do not exist* n *closed sets* $A_1, A_2, \ldots A_n \subseteq X$ *such that*

$$A_i \cap \tau(A_i) = \emptyset \quad \text{and} \quad X = \bigcup_{i=1}^{n} (A_i \cup \tau(A_i)).$$

Proof. The equivalence of all the above versions, with the exception of (4) and (5), is the content of [23] Exercise 5, p. 102 (explicitly for (1), (6), (8), (10) and implicitly for the (2), (3), (7), (9)). It is also mentioned here that the proofs can be found in [34], for X paracompact (and any CW-complex is paracompact). So, to complete the proof, it suffices to show (4) $\Longrightarrow$ (5) $\Longrightarrow$ (3) $\Longrightarrow$ (4).

(4) $\Longrightarrow$ (5) Assume to the contrary that $\gamma \simeq i\beta$ for some $\beta : W \to \mathbb{R}P^{n-1}$. Then $\beta^* i^*(y) = \gamma^*(y) = x$, contrary to (4).

(5) $\Longrightarrow$ (3) $\equiv$ (1) Assume (3) is false, so there is a $\mathbb{Z}_2$-equivariant map $F : X \to S^{n-1}$. Then, dividing out by the action of $\mathbb{Z}_2$, and calling β the induced map of F, one has a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{F} & S^{n-1} & \hookrightarrow & S^\infty \\ p\big\downarrow & & \kappa\big\downarrow & & \kappa\big\downarrow \\ W & \xrightarrow{\beta} & \mathbb{R}P^{n-1} & \xhookrightarrow{i} & \mathbb{R}P^\infty \end{array} .$$

It follows that $i \circ \beta := \gamma$ is a classifying map for the principal bundle and that it compresses to $\mathbb{R}P^{n-1}$, contrary to (5).

(1) $\equiv$ (3) $\Longrightarrow$ (4) Assuming (4) false, we have $\gamma := i \circ \beta : W \to \mathbb{R}P^\infty$ satisfies $\gamma^*(y) = \beta^* i^*(y) = x$, so γ is a classifying map for the principal bundle. Then there exists $\Gamma : X \to S^\infty$ and a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Gamma} & S^\infty \\ p\big\downarrow & & \kappa\big\downarrow \\ W & \xrightarrow{\gamma} & \mathbb{R}P^\infty \end{array} .$$

Since $\mathrm{Im}(\gamma) \subseteq \mathbb{R}P^{n-1}$, it follows that $\mathrm{Im}(\Gamma) \subseteq S^{n-1}$, *i.e.* Γ corestricts to a $\mathbb{Z}_2$-map $F : X \to S^{n-1}$ and this shows (3) is false. $\square$

Corollary 2.3. *Let τ_1, τ_2 be two free involutions on X with corresponding identification spaces W_1, W_2 and classifying maps γ_1, γ_2. Suppose the respective characteristic classes $x_1 \in H^1(W_1; \mathbb{Z}_2)$, $x_2 \in H^1(W_2; \mathbb{Z}_2)$ are equivalent in the sense that there exists a homotopy equivalence $h : W_1 \to W_2$ with $h^*(x_2) = x_1$. The Borsuk-Ulam theorem then holds for (X, τ_1, n) if and only if it holds for (X, τ_2, n).*

Proof. Since $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}_2, 1)$, the hypotheses imply that $\gamma_1 \simeq \gamma_2 \circ h$. Thus γ_1 compresses to $\mathbb{R}P^{n-1}$ if and only if $\gamma_2 \circ h$ does. But clearly $\gamma_2 \circ h$ compresses to $\mathbb{R}P^{n-1}$ if and only γ_2 does, since h is a homotopy equivalence. The result now follows from the equivalence of (1) and (5) in Proposition 2.2. □

In the next lemma we will make a reduction of the problem of finding all n such that (X, τ, n) is a Borsuk-Ulam triple, for given (X, τ).

Lemma 2.4. *Let (X, τ) be as above, n a positive integer and $m = \dim(X)$.*

(1) *If the Borsuk-Ulam theorem holds for (X, τ, n), then it also holds for the triple $(X, \tau, n-1)$.*

(2) *The Borsuk-Ulam theorem always holds for $(X, \tau, 1)$.*

(3) *The Borsuk-Ulam theorem never holds for $(X, \tau, m+1)$.*

Proof. Part (1) is clear, simply by composing a map $f : X \to R^{n-1}$ with the standard inclusion $\mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$ into the first $n-1$ coordinates.

Part (2) follows from the Proposition 2.2 above (using the equivalence of (1) with (3)), the fact that there is no surjective map from $X \to S^0$ since X is connected, and finally the fact that any equivariant map $X \to S^0$ would clearly have to be surjective.

Part (3) also follows from Proposition 2.2, using the equivalence of (1) and (6), together with the fact that the classifying map γ must compress to $\mathbb{R}P^m$ by cellular approximation. □

After this lemma, we can now say that the original problem is equivalent to the following reformulation: Given (X, τ) find the highest integer n such that the Borsuk-Ulam theorem holds for (X, τ, n). Furthermore, we know from the lemma that $1 \leq n \leq m = \dim(X)$. From Proposition 2.2 it is clear that this largest n

equals $\mathrm{ind}_{\mathbb{Z}_2}(X,\tau)$, so one also has the inequality $1 \le \mathrm{ind}_{\mathbb{Z}_2}(X,\tau) \le \dim(X)$. In Section 4 we shall take $m = 2$, so in this case the only possibilities are $n = 1, 2$. In Section 5 we shall take $m = 3$, so here the only possibilities are $n = 1, 2, 3$.

3 The main results

As before, X will denote a connected m-dimensional CW-complex with fixed point free involution τ, $W = X/\tau$, and $\gamma : W \to \mathbb{R}P^\infty$, $y \in H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$, $x = \gamma^*(y) \in H^1(W;\mathbb{Z}_2)$ have the same meanings as in Section 2. Of course $\mathbb{R}P^1 \cong S^1$, but it will be convenient to write it as $\mathbb{R}P^1$ and also write $i : \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty$ for the inclusion. In this section, we now give simple and computable criteria to decide when the triple (X, τ, n) has the Borsuk-Ulam property for $n = 2$ and for $n = m$.

Theorem 3.1. *Let (X, τ) be as above and γ the classifying map of the principal bundle $X \to W$. Let $\rho : H^1(\ ;\mathbb{Z}) \to H^1(\ ;\mathbb{Z}_2)$ denote the usual coefficient homomorphism, and let $i : \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty$ denote the inclusion. Then the following are equivalent:*

(a) *the Borsuk-Ulam theorem fails for (X, τ, n), $n \ge 2$,*

(b) *there exists a homomorphism $\beta : \pi_1(W) \to \pi_1(\mathbb{R}P^1) \approx \mathbb{Z}$ satisfying $i_\#\beta = \gamma_\# : \pi_1(W) \to \pi_1(\mathbb{R}P^\infty) \approx \mathbb{Z}_2$,*

(c) $x \in \mathrm{Im}(\rho) \subseteq H^1(W;\mathbb{Z}_2)$.

Furthermore, any of (a), (b), (c) *imply* $x^2 = 0$.

Proof. (a) $\Longrightarrow$ (b) From the equivalence of (1) and (4) in Proposition (2.2) we have (a) implies that the classifying map γ compresses to $\mathbb{R}P^1$. So there exists $f : W \to \mathbb{R}P^1$ with $i \circ f \simeq \gamma$. Simply take $\beta = f_\#$.

(b) $\Longrightarrow$ (c) We are given $i_\#\beta = \gamma_\# : \pi_1(W) \to \pi_1(\mathbb{R}P^\infty)$. Let $f\colon W \to \mathbb{R}P^1$ such that $f_\# = \beta$. Such an f exists since

$$\mathrm{hom}(\pi_1(W),\mathbb{Z}) \approx \mathrm{hom}(H_1(W),\mathbb{Z}) \approx H^1(W;\mathbb{Z}) \approx [W;\mathbb{R}P^1].$$

Now $(i \circ f)_\# = i_\# f_\# = \gamma_\#$ so applying a similar sequence of isomorphisms

$$\mathrm{hom}(\pi_1(W),\mathbb{Z}_2) \approx \mathrm{hom}(H_1(W),\mathbb{Z}_2) \approx H^1(W;\mathbb{Z}_2) \approx [W;\mathbb{R}P^\infty]$$

shows that $i \circ f \simeq \gamma$. Hence $x = \gamma^*(y) = f^*i^*(y)$. But $i^*(y) \in \mathrm{Im}(\rho : H^1(\mathbb{R}P^1;\mathbb{Z}) \to H^1(\mathbb{R}P^1;\mathbb{Z}_2))$, and using the naturality of ρ then shows that also $x = f^*i^*(y) \in \mathrm{Im}(\rho)$.

(c) $\Longrightarrow$ (a) We are given $x = \rho(\xi)$ for some $\xi \in H^1(W;\mathbb{Z})$. Now $H^1(W;\mathbb{Z}) \approx [W, K(\mathbb{Z},1)] \approx [W, \mathbb{R}P^1]$. Hence there exists a map $f : W \to \mathbb{R}P^1$ such that $f^*(\eta_0) = \xi$, where η_0 generates $H^1(\mathbb{R}P^1;\mathbb{Z})$. Also $\rho(\eta_0) = \eta$ where η generates $H^1(\mathbb{R}P^1;\mathbb{Z}_2)$. We now obtain

$$(i \circ f)^*(y) = f^*(i^*(y)) = f^*(\eta) = f^*(\rho(\eta_0)) = \rho f^*(\eta_0) = \rho(\xi) = x.$$

Thus $i \circ f \simeq \gamma$ and γ compresses to $\mathbb{R}P^1$. From the equivalence of (1) and (4) in Proposition (2.2) the Borsuk-Ulam theorem does not hold for $(X, \tau, 2)$. From Lemma (2.4), this result implies (a).

Finally, as noted in the first part of the above proof, (a) implies that γ compresses to $\mathbb{R}P^1$. Since any cup-square in $\mathbb{R}P^1$ equals 0 by dimensional reasons, it follows by naturality that $x^2 = 0$. □

A partial converse to the last part of Theorem 3.1 is now given.

Corollary 3.2. *If $x \in H^1(W;\mathbb{Z}_2)$, $x^2 = 0$, and also $2H_1(W;\mathbb{Z}) = 0$, then each of* (a)*,* (b)*,* (c) *hold.*

Proof. The condition $x^2 = 0$ is the same as $\mathrm{Sq}^1(x) = \beta(x) = 0$, where β is the mod 2 Bockstein, and this in turn is equivalent to $x \in \mathrm{Im}(\rho')$, where ρ' is the coefficient homomorphism induced by $\mathbb{Z}_4 \twoheadrightarrow \mathbb{Z}_2$. But it is easy to see that the remaining hypothesis $2H_1(W;\mathbb{Z}) = 0$ implies that any $\mathbb{Z}_4$ cohomology class (in dimension 1) is in fact an integral cohomology class. Thus $\mathrm{Im}(\rho') = \mathrm{Im}(\rho)$ in this situation, so $x \in \mathrm{Im}(\rho)$. □

The next corollary applies to the case when $\pi_1(W)$ is finite.

Corollary 3.3. *If $\pi_1(W)$ is finite, then $(X, \tau, 2)$ is always a Borsuk-Ulam triple.*

Proof. In this case $H_1(W;\mathbb{Z}) = (\pi_1(W))_{ab}$ is also finite, whence $H^1(W;\mathbb{Z}) \approx \hom(H_1(W;\mathbb{Z}),\mathbb{Z}) = 0$, i.e. $\mathrm{Im}(\rho) = 0$. Now apply (a) $\equiv$ (c) in Theorem 3.1. □

Now we assume that X is also an m-dimensional manifold, then so is W.

Theorem 3.4. *Using the above notations, the Borsuk-Ulam theorem holds for the triple* (X, τ, m) *if and only if the m-fold cup product* $x^m \neq 0$.

Proof. Suppose that x satisfies $x^m \neq 0$ and denote by $y_1 = i^*(y)$ the generator of $H^1(\mathbb{R}P^{m-1}; \mathbb{Z}_2)$. Then we claim that (4) of Proposition (2.2) holds. Otherwise we have a map $f : W \to \mathbb{R}P^{m-1}$ such that $f^*(y_1) = x$. But this is a contradiction since $y_1^m = 0$ implies $x^m = 0$, and the implication follows.

Conversely, suppose that (4) holds. We claim that $x^m \neq 0$. Suppose to the contrary $x^m = 0$. We have already seen that γ factors (up to homotopy) through $\mathbb{R}P^m$, i.e. $\gamma \simeq i \circ f$ for some $f : W \to \mathbb{R}P^m$ and $i : \mathbb{R}P^m \hookrightarrow \mathbb{R}P^\infty$. We shall now denote the generator of $H^1(\mathbb{R}P^m; \mathbb{Z}_2) \approx \mathbb{Z}_2$ by $y_1 = i^*(y)$.

The remainder of this argument will closely follow ideas and notation from both Epstein [9] and Olum [25]. Because $x^m = 0$, the degree of f with $\mathbb{Z}_2$-coefficients is zero, which in Epstein's notation says $a(f, 2) = 0$. Again, following Epstein, we write $A(f)$ for the "absolute degree" of f (this is the same, at least up to absolute value, as the "Absolutgrad" of Hopf [17], [18], and is exactly equal to what Olum calls grd(f), the "group-ring degree"). Again, using Olum's notation, deg(f) will denote the "twisted degree" of f, and $\theta = f_\# : \pi_1(W) \to \pi_1(\mathbb{R}P^m) \approx \mathbb{Z}_2$. All these degrees are congruent mod 2 ([25] p. 478) as well as to $a(f, 2)$ mod 2 ([9] Theorem 3.1). So in our situation, in all cases, both $A(f)$ and deg(f) are congruent to zero mod 2. Also, in our case, $\gamma_\# : \pi_1(W) \to \pi_1(\mathbb{R}P^\infty)$ is surjective (as remarked in §2), so the index $j = [\pi_1 \mathbb{R}P^m : \gamma_\#(\pi_1 W)] = 1$.

Once again, following [25] p. 475 or [9] p. 371, there are three possibilities for θ, as follows.

- If θ has Class I, then by Olum's Theorem IIa there exists $g : W \to \mathbb{R}P^m$ such that deg$(g) = 0$, $g_\# = f_\#$, and also deg$(g) = A(g)$ [25] p. 478. Using the isomorphism hom$(\pi_1(W), \mathbb{Z}_2) \approx H^1(W; \mathbb{Z}_2)$ (as in the proof of Theorem 3.1), we see that the condition $g_\# = f_\#$, which implies $(i \circ g)_\# = i_\# g_\# = i_\# f_\# = (i \circ f)_\#$, guarantees that $g^* i^* = f^* i* : H^1(\mathbb{R}P^\infty; \mathbb{Z}_2) \to H^1(W; \mathbb{Z}_2)$. This implies that x is also the characteristic class for the induced bundle of $i \circ g$, for we now have $g^*(y_1) = g^* i^*(y) = f^* i^*(y) = \gamma^*(y) = x$.
- If θ has Class II, then Olum's Theorem VIII implies that deg$(f) = 0$ with no further hypotheses, and also once again deg$(f) = A(f)$ by [25] p. 478.
- Finally, for θ of Class III, let us first note that in [9] (Diagram (1.3) and

the remarks following this diagram), the covering $\overline{N} \to N$ has j sheets (N, the codomain, corresponds to $\mathbb{R}P^m$ in the present work). Since the index $j = 1$ in our situation, it follows that $\overline{N} = N$ and hence $\overline{f} = f$. So following [9] p. 371, the Class III case, we have $0 = a(f, 2) = a(\overline{f}, 2) = A(f)$.

In any of the three cases, then, there is a map $g : W \to \mathbb{R}P^m$ with $A(g) = 0$ and $g^*(y_1) = x$ (indeed $g = f$ in the second and third cases). In this situation, using a theorem of Hopf (cf. [17], [18], also see [9], Theorem 4.1, for a more modern proof), the map g can be deformed by a homotopy to a map $h : W \to \mathbb{R}P^m$ such that the geometric degree (cf. [9], p. 372) of h equals its absolute degree, which in turn equals the absolute degree of g, namely 0. Therefore h, having geometric degree 0, is not surjective. Since $\mathbb{R}P^m$, with its usual CW-structure, has only a single m-cell, this implies that h can be deformed to a map that factors through $\mathbb{R}P^{m-1}$. Finally, observe that $i \circ h \simeq \gamma$, since $(i \circ h)^*(y) = (i \circ g)^*(y) = g^*(y_1) = x = \gamma^*(y)$. □

4 Applications to surfaces

In this brief section we shall see that when $m = 2$ and $X = M$ is a closed (connected) surface, the conditions for $(M, \tau, 2)$ to be a Borsuk-Ulam triple, which now come from both Theorem 3.1 and Theorem 3.4, coincide. A proposition will also be proved giving necessary and sufficient conditions for $(M, \tau, 2)$ to be a Borsuk-Ulam triple in terms of the orientability of the quotient surface $N := W = M/\tau$ and its $\mathbb{Z}_2$ cohomology. As observed in Section 2, only $(M, \tau, 1)$ or $(M, \tau, 2)$ can be Borsuk-Ulam triples in this situation. For further details on the 2-dimensional case cf. [11].

Proposition 4.1. *With M and $N = M/\tau$ closed surfaces as above, and x having the same meaning as in §2 and §3, the following are equivalent:*

(a) *the Borsuk-Ulam theorem holds for $(M, \tau, 2)$,*

(b) $x^2 \neq 0 \in H^2(N; \mathbb{Z}_2)$,

(c) $x \notin \mathrm{Im}(\rho : H^1(N; \mathbb{Z}) \to H^1(N; \mathbb{Z}_2))$.

Proof. The equivalences follow from the appropriate equivalences in Theorems 3.1 and 3.4, as well as Corollary 3.2 (which applies here since $2H_1(N;\mathbb{Z})$ contains no 2-torsion for any surface N). □

For the next proposition knowledge of the cohomology ring of a closed surface with $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients is needed, a good reference being [13] pp. 207-208. In particular, for $N = \mathbb{R}P^2 \# \underset{g}{\cdots} \# \mathbb{R}P^2$ non-orientable of genus g, one has $H^1(N;\mathbb{Z}_2) \approx \mathbb{Z}_2 \oplus \underset{g}{\cdots} \oplus \mathbb{Z}_2$ with generators $\alpha_1, ..., \alpha_g$ satisfying $\alpha_i^2 \neq 0$ and $\alpha_i \alpha_j = 0,\ i \neq j$.

Proposition 4.2. *Again, with M and $N = M/\tau$ closed surfaces, the Borsuk-Ulam theorem holds for $(M, \tau, 2)$ if and only if N is non-orientable and x is the sum of an odd number of α_i.*

Proof. If N is orientable then it is known that $\rho : H^1(N;\mathbb{Z}) \to H^1(N;\mathbb{Z}_2)$ is surjective. The equivalence of (a) and (c) in Proposition 4.1 above now shows that $(M, \tau, 2)$ is not a Borsuk-Ulam triple. For N non-orientable, the result is obvious from the equivalence of (a) and (b) in Proposition 4.1, from the description given above of $H^*(N;\mathbb{Z}_2)$, and finally from the mod 2 binomial theorem $(a+b)^2 = a^2 + b^2$. □

5 Applications to Seifert manifolds

The Seifert manifolds considered here are the 3-dimensional manifolds first introduced by Seifert [29] in 1933. They constitute an important class of 3-manifolds. In addition to Seifert's original paper, there are many more references such as the books by Orlik [26], Hempel [15], Jaco [19], and the papers of Scott [28] and of Jaco and Shalen [20]. The following description of the (closed) Seifert manifolds will therefore be brief, and is mainly intended so the notations used are clear. The fundamental group $\pi_1(M) := G$ of a Seifert manifold M was calculated in [29]. More recently, in [6], [7], [8], the cohomology rings $H^*(M;A)$ have been determined for orientable Seifert manifolds with infinite fundamental group G, and for various coefficients A (usually A is taken to be $\mathbb{Z}_p$, p prime, or $\mathbb{Z}$, as trivial $\mathbb{Z}G$-modules). In [33] the cohomology rings of the Seifert manifolds with G finite are calculated, such Seifert manifolds are all orientable. And in the preprint [2] the cohomology rings of all Seifert manifolds, both orientable and non-orientable, are determined.

As mentioned above, the manifolds that nowadays are called Seifert manifolds were first defined in [29], and were called "gefaserte Räume" or "fibred spaces" by Seifert. Such a manifold is first of all a closed 3-manifold that is a disjoint union of circles, called fibres. Identifying each fibre to a point gives an identification map $p : M \twoheadrightarrow V$, with the orbit space V being a closed surface. As usual, then, one has $V \cong T^2 \# \underset{g}{\cdots} \# T^2$ if orientable and $V \cong \mathbb{R}P^2 \# \underset{g}{\cdots} \# \mathbb{R}P^2$ if non-orientable, g being called the genus with $g \geq 0$ in the orientable case (for $g = 0$, V is the 2-sphere S^2) and $g \geq 1$ in the non-orientable case. The surface V is called the *base* (or *base surface*) of M. Furthermore, there are $r \geq 0$ points $x_1, \ldots, x_r \in V$ such that $p^{-1}(x_j)$ are called singular fibres. They have the following property: there is a closed disc neighbourhood $B_{x_j} := B_j$ with $p^{-1}(B_j)$ a "twisted" solid torus with twisting data given by a pair of coprime natural numbers (a_j, b_j), with $0 < b_j < a_j$. For each $y \in B_j,\ y \neq x_j,\ p^{-1}(y)$ is an (a_j, b_j) torus knot that twists a_j times in the direction parallel to the central longitudinal fibre $p^{-1}(x_j)$, and b_j times about this fibre in the meridian direction.

Now we consider $M \setminus \bigcup (p^{-1}(\mathrm{Int}(B_j))$ and we fill in each boundary component with an ordinary solid torus. Then we obtain a locally trivial S^1-fibration. The obstruction to this being a trivial S^1-fibration is an integer e called the Euler number. All this data is summarized by the notation $M = (\Xi, \xi, g \,|\, e; (a_1, b_1), \ldots, (a_r, b_r))$. Here Ξ takes the values O, N and describes the orientability (resp. non-orientability) of M. Similarly, ξ takes the values o, n and describes the same for the base V, while g equals the genus of V. The remaining data has already been explained.

In this section we shall first give an example of a Seifert manifold which does not admit a free involution, and then give examples of triples (M, τ, n) with both M and N Seifert manifolds which illustrate the various possibilities for the validity of the Borsuk-Ulam theorem, where the maximum possible value for n can equal $1, 2$, or 3. The calculations of the cohomology rings of the various manifolds N that occur, as described in the first paragraph of this section, will be used, in conjunction with the theorems in §3.

The collection of all finite groups G that can act freely and orthogonally on S^3 (necessarily $G \subset SO(4)$) was given in 1925 by Hopf [16] and by Threlfall and Seifert in 1931 [31] and 1933 [32]. The resulting family of groups was succinctly listed by Milnor [24] in 1957. Using his notation, these groups are C_n; $B_{2^k(2n+1)}, k \geq 2, n \geq 1$;

$Q_{4n},\ n \geq 1; P_{24}; P_{48}; P_{120}; P'_{8\cdot 3^k}, k \geq 1$, as well as the product of any of these groups with a cyclic group of relatively prime order. Here C_n, Q_{4n} are respectively the cyclic and generalized quaternion groups, and the subscript always denotes the order of the group (we use $B_{2^k(2n+1)}$ instead of Milnor's $D_{2^k(2n+1)}$ to avoid confusion since D_{2n} is usually used for the dihedral groups). Since the work of Perelman [27] in 2003 and the subsequent work, it is now known that this is the complete list of all finite fundamental groups of 3-manifolds, as well as all finite groups that admit a free action on S^3. All the orbit spaces S^3/G are Seifert manifolds, and are called spherical space forms. Shorter versions of the original Hopf, Threlfall-Seifert proofs, are now available thanks to Hattori [14] and Orlik [26]. In particular, the spherical space form S^3/P_{120} is the famous Poincaré sphere, which we shall denote by P and will be the subject of our first example. As a Seifert manifold $P = (O, o, 0 \,|\, -1; (2,1), (3,1), (5,1))$.

Example 5.1. *The Poincaré sphere P is a Seifert manifold which does not admit a free involution, and thus cannot be taken as the manifold M for any (M, τ).*

Proof. Suppose P_{120} is an index 2 subgroup of any group G. Then $G/P_{120} \approx \mathbb{Z}_2$ is abelian, so it follows that $[G,G] \subseteq P_{120}$. But this gives $P_{120} = [P_{120}, P_{120}] \subseteq [G,G] \subseteq P_{120}$, from which it follows that $[G,G] = P_{120}$ and hence $G_{ab} \approx \mathbb{Z}_2$. Now the only groups of order 240 on the list are: the cyclic group C_{240}, the quaternionic group Q_{240}, $B_{24(15)}$, and $H \times C_{2n+1}$ with $(2n+1)|H| = 240$. The abelianizations of these groups are well known and none is $\mathbb{Z}_2$, so it follows that P_{120} cannot be an index 2 subgroup of any of them. □

Remark 5.2. (a) The above proof can also be given without invoking the work of Perelman on the Poincaré Conjecture. Namely, one must then also consider the family of groups $Q(8n, k, l)$ (cf. [24]), and the same proof goes through with almost no extra work, since the abelianization of any of these groups is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(b) It is interesting to note that P also cannot fulfill the rôle of the manifold $N = M/\mathbb{Z}_2$. This can be seen by showing that $G := P_{120}$ has no subgroup H of index 2. For if such H existed, then it would necessarily be a normal subgroup which gives a surjection $G \twoheadrightarrow G/H \approx \mathbb{Z}_2$. And this is impossible since G, being perfect, has $G_{ab} = \{0\}$.

Now we give examples where the Borsuk-Ulam theorem does not hold for $(M, \tau, 2)$, so that only $n = 1$ is possible.

Example 5.3. Consider the Seifert manifold $N = S^1 \times V = (O, o, g \,|\, 0)$, where V is any closed orientable surface of genus $g \geq 0$, and let M be any connected two-fold covering N, given by a surjection $\theta : \pi_1(N) \approx \mathbb{Z} \times \pi_1(V) \twoheadrightarrow \mathbb{Z}_2$. Since $\rho : H^1(N;\mathbb{Z}) \to H^1(N;\mathbb{Z}_2)$ is surjective in this situation, condition (c) of Theorem 3.1 is automatically satisfied, so by Theorem 3.1 (a) only $n = 1$ is possible.

We conclude with three further examples. In Example 5.4 the Borsuk-Ulam theorem does not hold for $(M, \tau, 3)$, does holds for $(M, \tau, 2)$, and $x^2 \neq 0$. Example 5.5 has the property that the Borsuk-Ulam theorem holds for $(M, \tau, 3)$, so (from Theorem 3.4) $x^3 \neq 0$ in this example. In Example 5.6 the Borsuk-Ulam theorem does not hold for $(M, \tau, 3)$, does hold for $(M, \tau, 2)$, and $x^2 = 0$. The examples are Seifert manifolds such that the quotient N is a prism manifold, which we explain in more detail in the following paragraph. Since these manifolds all have finite fundamental group, $(M, \tau, 2)$ is a Borsuk-Ulam triple by Corollary 3.3. The answers as to when $(M, \tau, 3)$ can also be a Borsuk-Ulam triple will be given in terms of the Seifert invariants of N.

Prism manifolds seem to have been first introduced in [32]. A good reference for prism manifolds and their fundamental groups is Orlik [26], p. 107. However, there are a couple of small notational errors there, see [33] (Section 5, comments to Tables 1, 2) for corrections. A prism manifold can admit two different Seifert structures, let us consider the prism manifold N given by the following Seifert invariants

$$N = (O, o, 0 \mid e; (2,1), (2,1), (a,b)).$$

According to [26] and [33] (where $a = a_3,\ b = b_3$ is used) ,

$$\pi_1(N) \approx \begin{cases} \mathbb{Z}_m \times Q_{4a}, & m \equiv 1, 2, 3 \pmod 4, \\ \mathbb{Z}_{m''} \times B_{2^{k+3}\cdot a}, & m \text{ even}, \end{cases}$$

where $m = |(e+1)a + b|$, and in addition when m is even a must be odd, and $m = 2^{k+1} \cdot m''$ with m'' odd, $k \geq 0$. We remark that the case $m \equiv 2 \pmod 4$ occurs twice in the above isomorphisms, which is due to the isomorphism of groups $Q_{4a} \approx B_{4a}$, a odd. This is likely the reason for the remark in [26] that "if $m = 2m'$

then necessarily m' is even," which has as consequence that the case $m \equiv 2 \pmod 4$ is in fact overlooked in [26].

From [29] we have the following presentation for this fundamental group:

$$G := \pi_1(N) = \langle s_1, s_2, s_3, h \mid s_1^2 h,\ s_2^2 h,\ s_3^a h^b,\ [s_j, h],\ 1 \le j \le 3,\ s_1 s_2 s_3 h^{-e} \rangle.$$

The calculations in [6], [7], and [8] can be used to determine the $\mathbb{Z}_2$-cohomology ring of N, even though those references require that the fundamental group of the Seifert manifold be infinite (which then implies the manifold is aspherical, i.e. a $K(G,1)$), as we now explain. When N is a spherical space form, so has finite fundamental group G, it is no longer aspherical. Nevertheless, because N has universal cover S^3 (or equally well because N is irreducible), $\pi_2(N) = 0$. This implies that N approximates an aspherical space in the sense that N can be taken as the 3-skeleton of a $K(G,1)$. The resulting inclusion map $N \hookrightarrow K(G,1)$ therefore induces a cohomology isomorphism in dimensions $0, 1, 2$ and a monomorphism in dimension 3, which suffices to determine the cohomology ring of N in terms of the group cohomology of $\pi_1(N)$, as calculated in the cited references. This method is also used in [30] and [33].

The following two examples will have a even, which implies that m is odd. The cohomology ring $H^*(N;\mathbb{Z}_2)$ still depends on whether $a \equiv 2 \pmod 4$ or $a \equiv 0 \pmod 4$. We shall consider the latter case. In each example we will also give the homomorphism $\theta : \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ that identifies with the class x under the isomorphisms $\hom(\pi_1(N), \mathbb{Z}_2) \approx \hom((\pi_1(N))_{ab}, \mathbb{Z}_2) \approx \hom(H_1(N;\mathbb{Z}), \mathbb{Z}_2) \approx H^1(N;\mathbb{Z}_2)$, since $\mathrm{Ker}(\theta)$ determines the manifold M that double covers N. We follow the notation and quote the results of [8]. One has $n_2 = 3$ (the number of a_i that are divisible by 2), whence $H^*(N;\mathbb{Z}_2) \approx \mathbb{Z}_2\{1, \alpha_2, \alpha_3, \beta_2, \beta_3, \gamma\}$ as a graded $\mathbb{Z}_2$ vector space, with α_i, β_i, γ respectively in dimensions $1, 2, 3$. Taking into account that now $\binom{a}{2} \equiv 0 \pmod 2$, the cup products are given by $\alpha_2^2 = \beta_3,\ \alpha_3^2 = \beta_2 + \beta_3 = \alpha_2\alpha_3,\ \alpha_i\beta_j = \delta_{ij}\gamma$, which is readily seen to imply $\alpha_2^3 = 0,\ \alpha_3^3 = (\alpha_2 + \alpha_3)^3 = \gamma$.

Example 5.4. Consider $N = (O, o, 0 \,|\, e; (2,1), (2,1), (a,b))$ with a divisible by 4, and choose $x = \alpha_2$. Then $x^2 \neq 0,\ x^3 = 0$, whence $(M, \tau, 2)$ is a Borsuk-Ulam triple but not $(M, \tau, 3)$. With a little extra work one can determine that here

$$\theta(s_1) = 1,\ \ \theta(s_2) = 1,\ \ \theta(s_3) = 0,\ \ \theta(h) = 0,$$

and furthermore that in this case M is the lens space $L(2am, q)$, where (cf. [26] pp. 99-100) choosing $u, v \in \mathbb{Z}$ such that $ua - v((2e+2)a+b) = 1$, one has $q = ua + vb$. Furthermore the Seifert invariants are $M = (O, o, 0 \,|\, 2e+2; (a, b), (a, b))$.

Example 5.5. Again with N as in Example 5.4, consider $x = \alpha_2 + \alpha_3$. Now $x^3 \neq 0$ so $(M, \tau, 3)$ is a Borsuk-Ulam triple. Here

$$\theta(s_1) = 0, \quad \theta(s_2) = 1, \quad \theta(s_3) = 1, \quad \theta(h) = 0.$$

In this case M is the prism manifold $M = (O, o, 0 \,|\, 2e+2; (2, 1), (2, 1), (a/2, b))$. Again, we omit the proof of the last two assertions.

We remark that when $a \equiv 2 \,(\mathrm{mod}\, 4)$, all $x^3 = 0$, $x \in H^1(N; \mathbb{Z}_2)$, so $(M, \tau, 2)$ are Borsuk-Ulam triples in these cases, but not $(M, \tau, 3)$. The case m even will now be considered, then a is necessarily odd.

Example 5.6. Consider $N = (O, o, 0 \,|\, e; (2, 1), (2, 1), (a, b))$ and suppose that $m = |(e+1)a + b|$ is divisible by 4, so a is necessarily odd. In this case $n_2 = 1$ so according to [8] $H^*(N; \mathbb{Z}_2) \approx \mathbb{Z}_2\{1, \alpha_2, , \beta_2, \gamma\}$ with $\alpha_2^2 = 0$. There is now only one choice for the characteristic class, namely $x = \alpha_2$, and we therefore have an example where $(M, \tau, 2)$ is a Borsuk-Ulam triple, $(M, \tau, 3)$ is not, and $x^2 = 0$. And again without proof,

$$\theta(s_1) = 1, \quad \theta(s_2) = 1, \quad \theta(s_3) = 0, \quad \theta(h) = 0,$$

and M is the same lens space as in Example 5.4 (with a different involution).

Of course many other examples of Seifert manifolds illustrating the above properties can be given, thanks to the knowledge of their mod-2 cohomology rings. Elementary examples which illustrate the cases 5.5, 5.6, can be obtained by taking N to be a lens space with fundamental group $\mathbb{Z}_2$, $\mathbb{Z}_4$ (e.g. $L(2, 1)$, $L(4, 1)$ respectively). In the case where the fundamental group is $\mathbb{Z}_4$, the Borsuk-Ulam theorem does not hold for $(M, \tau, 3)$ while $(M, \tau, 2)$ is a Borsuk-Ulam triple. In this case also $x^2 = 0$, since x is the mod-2 reduction of a mod-4 cohomology class. The first case corresponds to the classical Borsuk-Ulam theorem since $L(2, 1) \cong \mathbb{R}P^3$.

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Three-dimensional spherical space forms

Akio Hattori

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Historical background and translators' remarks

In this section the translators will give some of the historical background to Hattori's paper [3], as well as mention some of the related developments that have transpired since the paper was written. The original paper gave only three references and these are indicated in the bibliography with a ★, all further references have been added by the translators (only the references after 1890 are given in the bibliography). The paper itself starts in Section 0, and the translation attempts to stay as close to the original as possible, with extra remarks added in a few places by the translators and placed in [ ].

The spherical space form problem has a history that can be thought of as dating back to Euclid and the Elements, especially Euclid's famous parallel postulate and the many failed attempts to prove it from the other axioms, through about 1800. The early nineteenth century saw the discovery of non-euclidean geometry by

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Lobachevsky, Bolyai, and Gauss, in particular what we now call hyperbolic geometry. In the middle of the nineteenth century Riemann discovered what we now call elliptic geometry, and set out the foundations of Riemannian geometry in his now famous Habilitationsschrift, delivered as a lecture in 1854 by Riemann at the University of Göttingen with a great deal of trepidation, Gauss being in the audience. The consistency of these geometries with Euclidean geometry was demonstrated in 1858 by Beltrami, who gave concrete Euclidean models of surfaces whose intrinsic geometry represents hyperbolic geometry or elliptic geometry. Of course the Euclidean plane is a model for Euclidean geometry. In terms of differential geometry, these are two dimensional manifolds with constant negative curvature (hyperbolic case), constant positive curvature (elliptic case), and zero Gaussian curvature (euclidean case). Besides revolutionizing geometry, these discoveries had major logical and philosophical implications.

The next mathematical steps were taken by Clifford in 1873, who discovered tori (i.e. compact surfaces) with constant curvature equal to zero. Klein [2] asked in 1890 whether one could classify all surfaces of constant curvature, and just one year later Killing [1] generalized the question to Riemannian manifolds of arbitrary dimension n and having constant curvature, thus giving birth to the "Clifford-Klein" space-form problem.

In 1925 Hopf introduced topological methods to study the space-form problem, and since then topology has been an indispensable tool towards the solution. He proved that the universal covering space of a Clifford-Klein space form must be either Euclidean space $\mathbb{R}^n$, hyperbolic space H^n, or the sphere S^n, corresponding of course to respectively zero, negative, or positive curvature. For the positive curvature case, to which we shall mainly restrict our attention henceforth, he also showed that the spherical space form problem in dimension $n-1$ was equivalent to finding the subgroups of $SO(n)$ that are finite and act freely on S^{n-1}, i.e. each non-identity transformation operates on S^{n-1} with no fixed points.

As an example of the efficacy of topological methods, we make a brief digression and prove that for any finite group G acting freely on S^{n-1}, the action of each element of G must be orientation preserving in case n is even, and in case n is odd G can only be the trivial group or the cyclic group C_2 of order 2. To prove this, let $g \in G \setminus \{e\}$, and consider the left translation ℓ_g, where $\ell_g(x) = gx$, $x \in S^{n-1}$. Note that ℓ_g

is fixed point free, hence its Lefschetz number $L(\ell_g) = 0$. Furthermore, being a homeomorphism of S^{n-1}, it has Brouwer degree $d_g := \deg(\ell_g) = \pm 1$. The definition of the Lefschetz number then gives $0 = 1 + (-1)^{n-1} d_g$. For n even this gives $d_g = +1$, as asserted. For n odd it gives $d_g = -1$ for all $g \neq e$. But this implies that $gh = e$ for any two non-identity elements $g, h \in G$, so G can only be trivial or C_2. We remark that this result is used at the start of §2 in Hattori's proof, and also that the proof given holds for topological actions, not just orthogonal ones.

The next development after Hopf's work was the two papers [10] of Threlfall and Seifert, in 1930 and 1933. The combined length of these two papers is over 110 pages, and they accomplish the solution to the Clifford-Klein space form problem in dimension 3, i.e. all finite subgroups G of $SO(4)$ that act freely on S^3 are found, and the resulting spherical space forms S^3/G are investigated. Needless to say these papers are not easy reading, and it was in 1957 that Milnor [5] first presented the list of these subgroups in an easily understood form. They are the cyclic groups C_n, the generalized quaternion groups Q_{4n}, the binary tetrahedral group P_{24}, binary octahedral P_{48}, binary icosahedral P_{120}, and two additional families $B_{2^k(2n+1)},\ k \geq 2,\ n \geq 1,\ \ P'_{8\cdot 3^k},\ k \geq 1$, as well as the product of any of these groups with a cyclic group of relatively prime order. In this notation the subscript always denotes the order of the group, and we remark that Milnor used the notation $D_{2^k(2n+1)}$ instead of $B_{2^k(2n+1)}$, which has caused some confusion since these groups are *not* dihedral groups. Milnor accomplished much more in this paper, regarding the topological spherical space form problem, but we shall restrict this survey to the original Clifford-Klein problem, i.e. to the orthogonal actions only. It is worth adding, however, that Vincent [11] had done some important work in 1947 towards the n-dimensional case, and that the methods of homological algebra and group theory were now becoming essential to the investigation.

This brings us to Hattori's work [3] in 1960, which remarkably gives a new derivation of the Threlfall-Seifert results in a three page paper! He accomplished this by taking advantage of quaternions and Lie group theory, in particular for the Lie group $SO(4)$. Since the English translation of this paper follows, we will move on to 1967 when the monumental book of Wolf [12] appeared, giving the full solution to the Clifford-Klein space form problem in all dimensions. Much of Wolf's work was based on that of Vincent, and he was also aware of Hattori's paper (pp. 226-227). In

1972 Orlik's book [6] "Seifert Manifolds" appeared, and a new derivation of the 3-dimensional spherical space forms list was given. Orlik seems to have been unaware of Hattori's paper. Indeed, on p. 103, he says "There seems to have been no significant progress in the problem of finding all 3-manifolds with finite fundamental group since the results of Hopf [4] and Seifert and Threlfall [10]. These articles are somewhat difficult to read." His new derivation also uses Lie group theory but makes very little use of quaternions, and is about ten pages long. It does, however, give a slightly more direct path towards obtaining the list (as given by Milnor). Important subsequent texts on the topology of 3-manifolds, such as those of Hempel, Jaco, Montesinos, P. Scott, or P. Gilkey's book on spherical space forms, also make no mention of the Hattori paper.

Discussion of the topological 3-dimensional spherical space form problem and the interesting developments since Milnor's paper would take us too far afield. However it is important to state that this problem was finally resolved in 2003 by Perelman [7]. His work on the Poincaré Conjecture also shows that any free finite group action on S^3 is in fact conjugate to an orthogonal action. Thus the Clifford-Klein spherical space forms in dimension 3 also constitute all topological spherical space forms in this dimension.

0 Introduction

Many examples of manifolds are obtained from homogeneous spaces and from fibre bundles. Spherical space forms, i.e. Riemannian manifolds having a sphere as universal cover [called "elliptical space forms" in Hattori's original paper [3]], are examples of this type having interesting properties that are easy to analyze, and are frequently used as concrete examples, e.g. lens spaces. The classification of n-dimensional spherical space forms is equivalent to the determination of the finite subgroups of the orthogonal group $O(n)$, such that each non-identity element acts as an isometry of S^{n-1} with no fixed points. In the 3-dimensional case, the complete classification was carried out in 1925 by Hopf [4] and in the early 1930's by Threlfall-Seifert [10], together with the study of the corresponding 3-manifolds. In this article we simplify the arguments used in [4] and [10] to prove the main result.

1 Preliminaries

We will identify the 3-sphere as the unit sphere in the space $\mathbb{H}$ of quaternions: $S^3 = \{a+bi+cj+dk \ : \ a,b,c,d \in \mathbb{R},\ a^2+b^2+c^2+d^2 = 1\}$. We write $a = \Re(a+bi+cj+dk)$ for the real part of a quaternion. For $q = a+bi+cj+dk \in S^3$, $q^{-1} = \overline{q} = a-bi-cj-dk$. The 2-sphere S^2 will be identified with the pure quaternions in $S^3 \ : \ S^2 = \{bi+cj+dk\} \subset S^3$.

Elements of S^3 will be denoted $q, q_1, q_2, ...$, and elements of S^2 denoted $q', q'_1, q'_2, \ldots$. For $q' \in S^2$ and any angle θ, $\cos\theta + \sin\theta \cdot q' \in S^3$. The 3-sphere is a simply connected Lie group with the product coming from the quaternions. The following properties (1.1)-(1.3) are immediate.

(1.1) For each fixed $q' \in S^2$, $\{\cos\theta + \sin\theta \cdot q' \ : \ \theta \in \mathbb{R}\}$ is a maximal torus of S^3 [calling this maximal torus $T_{q'}$, one has $q' \in T_{q'}$ and for any $q \in S^3$, $q \neq \pm 1$, one has uniquely $q = \cos\theta + \sin\theta\ q'$, $q' \in S^2$ and $T_q = T_{q'}$, where T_q is the maximal torus containing q].

(1.2) The centre of S^3 is the subgroup $\{+1, -1\}$.

(1.3) Left translation of S^3 by a fixed $q \in S^3$ is an isometry, similarly for right translation by q.

From these properties and using the fact that maximal tori of a Lie group are conjugate, we obtain the following further properties.

(1.4) $\Re(q_1) = \Re(qq_1q^{-1})$, so in particular the conjugation map $q_1 \mapsto qq_1q^{-1}$ leaves S^2 invariant.

(1.5) If $q_1qq_2^{-1} = q$, then $\Re(q_1) = \Re(q_2)$. Conversely, if $\Re(q_1) = \Re(q_2)$, then there exists a q such that $q_1qq_2^{-1} = q$.

(1.6) Defining the homomorphism $\phi : S^3 \times S^3 \to SO(4)$ by $\phi(q_1,q_2)(q) = q_1qq_2^{-1}$, ϕ is surjective with $\mathrm{Ker}(\phi) = \{(1,1),(-1,-1)\}$. Thus ϕ is a double cover and is the projection map of the universal cover $S^3 \times S^3$ of $SO(4)$ (cf. [9] §22.4).

(1.7) Defining the homomorphism $\psi : S^3 \to SO(3)$ by $\psi(q)(r') = qr'q^{-1}$ for any pure quaternion $r' \in S^2$, ψ is a double cover and is the projection map of the universal cover S^3 of $SO(3)$ (cf. [9] §22.3) [as a rotation, letting $q = \cos\theta + \sin\theta\ q'$ as in (1.1), $\psi(q)$ is the rotation of $\mathbb{R}^3$ having axis of rotation q' and angle of rotation 2θ about this axis (again where $q \neq \pm 1$)] .

2 The finite subgroups of $SO(4)$

The elements of the isometry group $O(4)$ of S^3 without fixed points belong to $SO(4)$, which is the connected component of $O(4)$ that contains the neutral element. Therefore the first priority is to obtain all the finite subgroups of $SO(4)$. These will in turn be the images of the finite subgroups of $S^3 \times S^3$ under the homomorphism ϕ. Consider the natural projections and inclusions $\pi_v : S^3 \times S^3 \to S^3$ and $i_v : S^3 \to S^3 \times S^3$, $v = 1, 2$ (for example, $\pi_1(q_1, q_2) = q_1, i_1(q) = (q, 1)$). Given a finite subgroup G of $S^3 \times S^3$, consider the finite subgroups $\overline{G}_v = \pi_v(G)$ and $G_v = i_v^{-1}(G)$. Then we have

(2.1) G_v is a normal subgroup of $\overline{G}_v$ and $G_1 \times G_2 \subset G \subset \overline{G}_1 \times \overline{G}_2$,

(2.2) $\pi_v : G/(G_1 \times G_2) \to \overline{G_v}/G_v$ is bijective.

By (2.1) and (2.2), we have

(2.3) Considering the finite subgroups $\overline{G}_v$ ($v = 1, 2$) of S^3 and their normal subgroups G_v ($v = 1, 2$) such that $\overline{G}_1/G_1 \approx \overline{G}_2/G_2$, we have a bijective correspondence between the 5-tuples $(\overline{G}_1, G_1, \overline{G}_2, G_2, f)$ and the finite subgroups G of $S^3 \times S^3$, where $f : \overline{G}_1/G_1 \to \overline{G}_2/G_2$ is an isomorphism. This correspondence is given by associating $H = \{(x, y) : (x, y) \in \overline{G}_1/G_1 \times \overline{G}_2/G_2\,, f(x) = y\}$ to $G = p^{-1}(H)$, where $p : \overline{G}_1 \times \overline{G}_2 \to \overline{G}_1/G_1 \times \overline{G}_2/G_2$ is the natural projection.

Because of (2.3), the problem is reduced to studying the finite subgroups of S^3. Moreover, the study of the conjugation relations among the finite subgroups of $S^3 \times S^3$, using the automorphisms of $S^3 \times S^3$, can be viewed as the analysis of the relations among the components of $(\overline{G}_1, G_1, \overline{G}_2, G_2, f)$ of (2.3), but we need not enter into the details.

3 The list of finite subgroups of $SO(4)$

The subgroups of S^3 are exhausted by the inverse images of subgroups of $SO(3)$ by ψ, and their subgroups. As is well known (see for instance [13]), the finite subgroups of $SO(3)$ are exactly the subgroups given by orthogonal transformations that map a regular polygon or a regular polyhedron in $\mathbb{R}^3$, with centre at the origin, to itself. The table of corresponding subgroups and geometric figures is as follows:

(3.1)

Cyclic group C_n Dihedral group D_{2n}	Regular polygon of n sides
Tetrahedral group T	Regular tetrahedron
Octahedral group O	Cube, regular octahedron
Icosahedral group I	Regular dodecahedron, regular icosahedron

It follows that the finite subgroups of S^3 are exactly the cyclic groups, $\psi^{-1}(D_{2n})$, $\psi^{-1}(T)$, $\psi^{-1}(O)$ and $\psi^{-1}(I)$ (up to conjugation). An element σ of $SO(3)$ is determined by the axis of rotation and the angle of rotation. On the other hand, by (1.1) and (1.7), for $q \in S^3$, the axis of rotation of $\psi(q) \in SO(3)$ is the line passing through the points q_1, q_2, which are the intersection points of the maximal torus containing q and the sphere S^2 [where $q = \cos\theta + \sin\theta \cdot q_1$ and $q_2 = -q_1$]. Using these facts and the correspondence in Table (3.1), it is easy to describe a representative of the conjugation classes of each finite subgroup of S^3. For example, the representative of the cyclic group of order m is

$$\{\cos\frac{2\pi}{m}v + \sin\frac{2\pi}{m}v \cdot i\,;\ v = 0, 1, \ldots, m-1\}.$$

In fact, the subgroup G of S^3 that fixes i is a cyclic group and the line through i and $-i$ is the rotation axis of the elements ($\neq e$) of G. All the subgroups of $\psi^{-1}(G)$ have the above form. Listing all the cyclic subgroups of $\psi^{-1}(I)$, we have

$$A_1, A_2, \ldots, A_{15}, B_1, B_2, \ldots, B_{10}, C_1, C_2, \ldots, C_6 \subset \psi^{-1}(I).$$

Denoting by $\{q\}$ the cyclic group generated by q, we have

(3.2) $$A_1 = \{i\}, A_2 = \{j\}, A_3 = \{k\},$$

$$A_4 = \left\{\frac{1}{2}\left(\frac{\sqrt{5}+1}{2}i + j + \frac{\sqrt{5}-1}{2}k\right)\right\}, \qquad A_5 = \left\{\frac{1}{2}\left(\frac{\sqrt{5}+1}{2}i + j - \frac{\sqrt{5}-1}{2}k\right)\right\},$$

$$A_6 = \left\{\frac{1}{2}\left(\frac{\sqrt{5}+1}{2}i - j + \frac{\sqrt{5}-1}{2}k\right)\right\}, \qquad A_7 = \left\{\frac{1}{2}\left(\frac{\sqrt{5}+1}{2}i - j - \frac{\sqrt{5}-1}{2}k\right)\right\},$$

$$A_8 = \left\{ \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}j + k + \frac{\sqrt{5}-1}{2}i \right) \right\}, \qquad A_9 = \left\{ \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}j + k - \frac{\sqrt{5}-1}{2}i \right) \right\},$$

$$A_{10} = \left\{ \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}j - k + \frac{\sqrt{5}-1}{2}i \right) \right\}, \qquad A_{11} = \left\{ \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}j - k - \frac{\sqrt{5}-1}{2}i \right) \right\},$$

$$A_{12} = \left\{ \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}k + i + \frac{\sqrt{5}-1}{2}j \right) \right\}, \qquad A_{13} = \left\{ \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}k + i - \frac{\sqrt{5}-1}{2}j \right) \right\},$$

$$A_{14} = \left\{ \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}k - i + \frac{\sqrt{5}-1}{2}j \right) \right\}, \qquad A_{15} = \left\{ \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}k - i - \frac{\sqrt{5}-1}{2}j \right) \right\},$$

$$B_1 = \left\{ \cos\frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \frac{1}{\sqrt{3}}(i+j+k) \right\}, \qquad B_2 = \left\{ \cos\frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \frac{1}{\sqrt{3}}(-i+j+k) \right\},$$

$$B_3 = \left\{ \cos\frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \frac{1}{\sqrt{3}}(i-j+k) \right\}, \qquad B_4 = \left\{ \cos\frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \frac{1}{\sqrt{3}}(i+j-k) \right\},$$

$$B_5 = \left\{ \cos\frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \frac{1}{\sqrt{3}}\left(\frac{\sqrt{5}+1}{2}i + \frac{\sqrt{5}-1}{2}j \right) \right\},$$

$$B_6 = \left\{ \cos\frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \frac{1}{\sqrt{3}}\left(\frac{\sqrt{5}+1}{2}i - \frac{\sqrt{5}-1}{2}j \right) \right\},$$

$$B_7 = \left\{ \cos\frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \frac{1}{\sqrt{3}}\left(\frac{\sqrt{5}+1}{2}j + \frac{\sqrt{5}-1}{2}k \right) \right\},$$

$$B_8 = \left\{ \cos\frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \frac{1}{\sqrt{3}}\left(\frac{\sqrt{5}+1}{2}j - \frac{\sqrt{5}-1}{2}k \right) \right\},$$

$$B_9 = \left\{ \cos\frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \frac{1}{\sqrt{3}}\left(\frac{\sqrt{5}+1}{2}k + \frac{\sqrt{5}-1}{2}i \right) \right\},$$

$$B_{10} = \left\{ \cos\frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \frac{1}{\sqrt{3}}\left(\frac{\sqrt{5}+1}{2}k - \frac{\sqrt{5}-1}{2}i \right) \right\},$$

$$C_1 = \left\{ \cos\frac{\pi}{5} + \sin\frac{\pi}{5} \cdot c\left(\frac{\sqrt{5}+1}{2}i + \frac{\sqrt{5}+3}{2}j \right) \right\},$$

$$C_2 = \left\{ \cos\frac{\pi}{5} + \sin\frac{\pi}{5} \cdot c\left(-\frac{\sqrt{5}+1}{2}i + \frac{\sqrt{5}+3}{2}j \right) \right\},$$

$$C_3 = \left\{ \cos\frac{\pi}{5} + \sin\frac{\pi}{5} \cdot c\left(\frac{\sqrt{5}+1}{2}j + \frac{\sqrt{5}+3}{2}k \right) \right\},$$

$$C_4 = \left\{ \cos\frac{\pi}{5} + \sin\frac{\pi}{5} \cdot c\left(-\frac{\sqrt{5}+1}{2}j + \frac{\sqrt{5}+3}{2}k \right) \right\},$$

$$C_5 = \left\{ \cos\frac{\pi}{5} + \sin\frac{\pi}{5} \cdot c\left(\frac{\sqrt{5}+1}{2}k + \frac{\sqrt{5}+3}{2}i\right)\right\},$$
$$C_6 = \left\{ \cos\frac{\pi}{5} + \sin\frac{\pi}{5} \cdot c\left(-\frac{\sqrt{5}+1}{2}k + \frac{\sqrt{5}+3}{2}i\right)\right\},$$

where $c = \frac{1}{\sqrt{5+2\sqrt{5}}}$.

4 Fixed points

For a finite subgroup G of $S^3 \times S^3$, we will obtain the condition that the elements of $\phi(G)$ different from the neutral element have no fixed points.

For $(q_1, q_2) \in S^3 \times S^3$, the necessary and sufficient condition that $\phi(q_1, q_2)$ has fixed points is, by (1.5), that $\Re(q_1) = \Re(q_2)$. From this fact we have

(4.1) If the elements of $\phi(G)$ do not have fixed points, then, when constructing $\overline{G_1}, \overline{G_2}$ as in §2, at least one of $\overline{G_1}$ or $\overline{G_2}$ is a cyclic group.

It can be verified by checking each case that, if neither $\overline{G_1}$ nor $\overline{G_2}$ is cyclic, then there exists an element $(q_1, q_2) \in G$ where q_1, q_2 have order 4. This concludes (4.1). If one of $\overline{G_1}, \overline{G_2}$ is cyclic, it is easy to verify whether or not $\phi(G)$ has fixed points. For instance, when $G = \overline{G_1} \times \overline{G_2}$ and $\overline{G_1}, \overline{G_2}$ are both cyclic, the necessary and sufficient condition that $\phi(G)$ has no fixed points is that the orders of $\overline{G_1}$ and of $\overline{G_2}$ are relatively prime.

If $\phi(G)$ has no fixed points, we can consider

$$F = \{q \in S^3; \rho(1, q) \leqq \rho(\phi(g) \cdot 1, g), g \in G, \phi(g) \neq e\}$$

as the (closed) fundamental domain, where ρ is the distance in S^3. For instance, let us suppose that

$$G = \overline{G_1} \times \overline{G_2}, \overline{G_1} = \{q_1\}, \overline{G_2} = \{q_2\}$$

$$q_1 = \cos\frac{2\pi}{m} + \sin\frac{2\pi}{m} \cdot i, q_2 = \cos\frac{2\pi}{n} + \sin\frac{2\pi}{n} \cdot i$$

with $(m,n)=1$. Then $\overline{G_v}$ is the cyclic group generated by q_v and $G \cong \phi(G)$ is a cyclic group of order mn.

For integers l_1, l_2 such that $-ml_2 + nl_1 = 1$, the orbit of 1 by G is generated by the powers

$$q_1^{l_1} q_2^{l_2} = \cos\frac{2\pi}{mn} + \sin\frac{2\pi}{mn}\cdot i\,.$$

If we put $q_+ = \cos\frac{2\pi}{mn} + \sin\frac{2\pi}{mn}\cdot i$, $q_- = \cos\frac{2\pi}{mn} - \sin\frac{2\pi}{mn}\cdot i$ and denote the intersection between the hyperplane passing through $0, j, k, q_+(q_-)$ and the sphere S^3 by $B_+(B_-)$, then F contains 1 in its interior and is the region bounded by B_+ and B_-. The intersection $B_+ \cap B_-$ (a great circle) is obtained as the intersection of the sphere S^3 with the plane that passes through j, k and it is orthogonal to $1, i$. Each element $g \in G$ maps $B_+ \cap B_-$ to itself and G acts on $B_+ \cap B_-$ as a rotation group of order mn.

It is easy to verify that the only element $\phi(g) \in \phi(G)$ satisfying $\phi(g)(B_- - B_+) \subset B_+ \cup B_-$ is $\phi(q_1^{l_1}, q_2^{l_2})$, and that the only element $\phi(g) \in \phi(G)$ satisfying $\phi(g)(B_+ - B_-) \subset B_+ \cup B_-$ is $\phi(q_1^{-l_1}, q_2^{-l_2})$. Moreover, for $q' \in B_+ \cap B_-$, we have $\phi(q_1^{l_1}, q_2^{l_2}) \cdot q' = q_1^{l_1} \cdot q' \cdot q_2^{-l_2} = q_1^{l_1} \cdot q_2^{l_2} \cdot q' = \left(\cos\frac{2(ml_2+nl_1)}{mn} + \sin\frac{2(ml_2+nl_1)}{mn}\cdot i\right)\cdot q'$. That is, $\phi(q_1^{l_1}, q_2^{l_2})$ gives rise to a rotation by angle $\frac{2(ml_2+nl_1)}{mn}\pi$ on the great circle $B_+ \cap B_-$. Writing $(n-m)^{-1}$ for the integer such that $(n-m)^{-1}\cdot(n-m) \equiv 1 \pmod{mn}$, we have $ml_2 + nl_1 \equiv (m+n)(n-m)^{-1} \pmod{mn}$.

By the above observation, we can conclude that the quotient space $G\backslash S^3$ is homeomorphic to the lens space $L(mn, (m+n)(n-m)^{-1})$ (for lens spaces, cf. the book [8] of Seifert and Threlfall, p. 210).

In the same way, we can prove that $\psi^{-1}(I)\times 1\backslash S^3$ is homeomorphic to the dodecahedral space defined in [8], p. 216, also known as the Poincaré sphere. Since the commutator subgroup of $\psi^{-1}(I)$ coincides with $\psi^{-1}(I)$, we have that $\pi(\psi^{-1}(I)\times 1\backslash S^3) \cong \psi^{-1}(I)$, $H_1(\psi^{-1}(I)\times 1\backslash S^3) = 0$. Thus $\psi^{-1}(I)\times 1\backslash S^3$ is a homology sphere that is not simply connected.

5 T-bundles

If $\phi(G)$ has no elements with fixed points, we may by (4.1) regard $\overline{G_2}$ as a cyclic group (replacing by the conjugate, if necessary). If we denote by T the maximal torus that contains $\overline{G_2}$, the actions of G and T on S^3 commute. Therefore, if $M = G\backslash S^3$, we have

(5.1) T acts on M (on the right).

(5.2) Moreover, each orbit is homeomorphic to T.

By (5.2), we know that the isotropy group of each point is a cyclic group. An orbit with isotropy group $\neq \{e\}$ is called exceptional.

(5.3) There exists a finite number of exceptional orbits.

The pair of a manifold M satisfying (5.1)-(5.3) and the maximal torus T acting on M is called a T-bundle in the generalized sense [also called a "Seifert bundle," cf. [6] p. 83]. The main theorem of [10] is the converse of (5.1)-(5.2), that is

(5.4) if (M, T) is a T-bundle in the generalized sense with M a closed orientable 3-manifold having finite fundamental group, then M is a spherical space form.

This can now be proved easily along the following lines.

a) The universal cover $\widetilde{M}$ of M has a natural structure as a T-bundle.

b) $(\widetilde{M}, T)$ is, in fact, a T-bundle (if it has an exceptional orbit, it will not be null homotopic as a closed curve).

c) $\widetilde{M}/T \cong M/T \cong S^2$.

d) By a), b) and c), $(\widetilde{M}, T)$ is a T-bundle over S^2 and $\pi_1(\widetilde{M}) = 0$. So, by the Classification Theorem of [9, §26.2], we have $\widetilde{M} = S^3$.

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On projective bundles over small covers (a survey)

Shintarô KUROKI

ABSTRACT. In this article we survey results of a joint paper of the author with Z. Lü which is devoted to studying constructions of projective bundles over small covers. In order to construct all projective bundles from certain basic projective bundles, we introduce a new operation which is a combinatorial adaptation of the fibre sum of fibre bundles. By using this operation, we give a topological characterization of all projective bundles over 2-dimensional small covers.

1 Introduction

In the toric topology, the following question asked by Masuda and Suh in [11] is still an open and interesting problem:

Problem 1 (Cohomological rigidity problem). *Let M and M' be two (quasi)toric manifolds. Are M and M' homeomorphic if $H^*(M) \cong H^*(M')$?*

Small covers are a real analogue of quasitoric manifolds introduced by Davis and Januszkiewicz in [4]. In the paper [10], Masuda gives counterexamples to cohomological rigidity of small covers, that is, he shows that the cohomology ring does not distinguish between small covers. He classified diffeomorphism types of height 2 (generalized) real Bott manifolds by using the KO-ring structure of real projective spaces, where the *height* 2 (generalized) real Bott manifold is the total space of projective

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bundle of the Whitney sum of some line bundles over real projective space, and he found counterexamples to cohomological rigidity in height 2 (generalized) real Bott manifolds. It follows that projective bundles over small covers are not determined by the cohomology ring only. In this article, we try to study topological types of such projective bundles from a different point of view, as compared to Masuda's approach. Our method is closely related to Orlik-Raymond's method of [16]. In this paper, Orlik and Raymond show that 4-dimensional simply connected torus manifolds, i.e., objects satisfying weaker conditions than the corresponding quasitoric manifolds, can be constructed from certain basic torus manifolds by using connected sums. Since small covers (resp. 2-torus manifolds) M^n are real analogues of the quasitoric manifolds (resp. torus manifolds), it seems reasonable to try to apply the Orlik-Raymond method of [16] to 2-dimensional small covers. The goal of this article is to adapt their method to projective bundles over 2-dimensional small covers, and introduce the following construction theorem of such projective bundles.

Theorem 1. *Let $P(\xi)$ be a projective bundle over a 2-dimensional small cover M^2. Then $P(\xi)$ can be constructed from projective bundles $P(\kappa)$ over $\mathbb{R}P^2$ and $P(\zeta)$ over T^2 by using the projective connected sum $\sharp^{\Delta^{k-1}}$.*

The organization of this paper is as follows. In Section 2, we recall basic properties of small covers. In Section 3, we present Masuda's counterexample to cohomological rigidity. In Section 4, we study the structure of projective bundles over small covers and introduce new characteristic functions. In Section 5, we recall the construction theorem of 2-dimensional small covers and show a topological classification of projective bundles over basic 2-dimensional small covers. In Section 6, we present our main theorem. Finally, in Section 7, as an appendix, we exhibit a topological classification of projective bundles over 1-dimensional small covers.

2 Basic properties of small covers

We first recall the definition of small covers and some basic facts.

A convex n-dimensional polytope is called *simple* if the number of its *facets* (that is, codimension-one faces) meeting at every vertex is equal to n. Let $\mathbb{Z}_2 = \{-1, 1\}$ be the 2-element group.

2.1 Definition of small covers

An n-dimensional closed smooth manifold M is called a *small cover* over a simple convex n-polytope P if M has a $(\mathbb{Z}_2)^n$-action such that

(a) the $(\mathbb{Z}_2)^n$-action is *locally standard*, i.e., locally the same as the standard $(\mathbb{Z}_2)^n$-action on $\mathbb{R}^n$, and

(b) its orbit space is homeomorphic to P; the corresponding orbit projection map $\pi : M \to P$ is constant on $(\mathbb{Z}_2)^n$-orbits and maps every rank k orbit (i.e., every orbit isomorphic to $(\mathbb{Z}_2)^k$) to an interior point of a k-dimensional face of the polytope P, $k = 0, \ldots, n$.

We can easily show that π sends $(\mathbb{Z}_2)^n$-fixed points in M to vertices of P by using the above condition (b). We often call P an *orbit polytope* of M. We will use the symbol $(M, (\mathbb{Z}_2)^n)$ to denote an n-dimensional small cover M with $(\mathbb{Z}_2)^n$-action.

Let us look at three examples of small covers.

Example 2.1. Let $\mathbb{R}P^n$ be the n-dimensional real projective space and $[x_0 : x_1 : \cdots : x_n]$ be an element of $\mathbb{R}P^n$ represented by the standard projective coordinates. We define a $(\mathbb{Z}_2)^n$-action as follows:

$$[x_0 : x_1 : \cdots : x_n] \xrightarrow{(t_1,\ldots,t_n)} [x_0 : t_1x_1 : \cdots : t_nx_n],$$

where $(t_1, \ldots, t_n) \in (\mathbb{Z}_2)^n$. Then, one can easily check that this action is locally standard and the orbit polytope is the n-dimensional simplex Δ^n, i.e., the above $(\mathbb{R}P^n, (\mathbb{Z}_2)^n)$ is a small cover.

Example 2.2. We define the $\mathbb{Z}_2$-action on $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$ by the antipodal action on the first S^1-factor and the complex conjugation on the second S^1-factor. Let $\mathcal{K}^2 = S^1 \times_{\mathbb{Z}_2} S^1$ be the orbit space of this action. Then, $\mathcal{K}^2$ is the 2-dimensional manifold with the following $(\mathbb{Z}_2)^2$-action:

$$[x_1 + \sqrt{-1}y_1, x_2 + \sqrt{-1}y_2] \xrightarrow{(t_1,t_2)} [x_1 + t_1\sqrt{-1}y_1, x_2 + t_2\sqrt{-1}y_2],$$

where $[x_1+\sqrt{-1}y_1, x_2+\sqrt{-1}y_2] \in \mathcal{K}^2$ and $(t_1, t_2) \in (\mathbb{Z}_2)^2$. Then, one can easily check that $(\mathcal{K}^2, (\mathbb{Z}_2)^2)$ is a small cover whose orbit polytope is the 2-dimensional square I^2, where $I = [0, 1]$ is the interval. Note that $\mathcal{K}^2$ is an S^1-bundle over S^1; more precisely, we see that $\mathcal{K}^2$ is diffeomorphic to the Klein bottle.

Example 2.3. Let M_i be an n_i-dimensional small cover with $(\mathbb{Z}_2)^{n_i}$-action and P_i be its orbit polytope. Then, the product of manifolds with actions

$$\left(\prod_{i=1}^{a} M_i, \prod_{i=1}^{a} (\mathbb{Z}_2)^{n_i}\right)$$

is a small cover whose orbit polytope is $\prod_{i=1}^{a} P_i$, where $a \in \mathbb{N}$ and $\dim \prod_{i=1}^{a} M_i = \sum_{i=1}^{a} n_i$.

2.2 Construction of small covers

From another point of view, for a given simple polytope P, the small cover M with orbit projection $\pi : M \to P$ can be reconstructed by using the *characteristic function* $\lambda : \mathcal{F} \to (\mathbb{Z}/2\mathbb{Z})^n$, where $\mathcal{F}$ is the set of all facets in P and $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. In this subsection, we review this construction (see [2, 4] for details).

We first recall the characteristic function of the small cover M over P. Due to the definition of a small cover $\pi : M \to P$, we have that $\pi^{-1}(\mathrm{int}(F^{n-1}))$ consists of $(n-1)$-rank orbits, in other words, the isotropy subgroup at $x \in \pi^{-1}(\mathrm{int}(F^{n-1}))$ is $K \subset (\mathbb{Z}_2)^n$ such that $K \cong \mathbb{Z}_2$, where $\mathrm{int}(F^{n-1})$ is the interior of the facet F^{n-1}. Hence, the isotropy subgroup at x is determined by a primitive vector $v \in (\mathbb{Z}/2\mathbb{Z})^n$ such that $(\textbf{-1})^v$ generates the subgroup K, where $(\textbf{-1})^v = ((-1)^{v_1}, \ldots, (-1)^{v_n})$ for $v = (v_1, \ldots, v_n) \in (\mathbb{Z}/2\mathbb{Z})^n$. In this way, we obtain a function λ from the set of facets of P, denoted by $\mathcal{F}$, to vectors in $(\mathbb{Z}/2\mathbb{Z})^n$. We call such $\lambda : \mathcal{F} \to (\mathbb{Z}/2\mathbb{Z})^n$ a *characteristic function* or *colouring* on P. We often describe λ as the $m \times n$-matrix $\Lambda = (\lambda(F_1) \cdots \lambda(F_m))$ for $\mathcal{F} = \{F_1, \ldots, F_m\}$, and we call this matrix a *characteristic matrix*. Since the $(\mathbb{Z}_2)^n$-action is locally standard, a characteristic function has the following property (called *the property* $(\star)$):

$(\star)$ if $F_1 \cap \cdots \cap F_n \neq \emptyset$ for $F_i \in \mathcal{F}$ $(i = 1, \ldots, n)$, then $\{\lambda(F_1), \ldots, \lambda(F_n)\}$ spans $(\mathbb{Z}/2\mathbb{Z})^n$.

It is interesting that one also can construct small covers by using a given n-dimensional simple convex polytope P and a characteristic function λ with the property $(\star)$. Next, we mention the construction of small covers by using P and λ. Let P be an n-dimensional simple convex polytope. Suppose that a characteristic function $\lambda : \mathcal{F} \to (\mathbb{Z}/2\mathbb{Z})^n$ which satisfies the above property $(\star)$ is defined on P. Small covers can be constructed from P and λ as the quotient space $(\mathbb{Z}_2)^n \times P/\sim$, where the symbol $\sim$ represents an equivalence relation on $(\mathbb{Z}_2)^n \times P$ defined as follows: $(t, x) \sim (t', y)$ if and only if $x = y \in P$ and

$$\begin{array}{ll} t = t' & \text{if } x \in \text{int}(P); \\ t^{-1}t' \in \langle(\textbf{-1})^{\lambda(F_1)}, \ldots, (\textbf{-1})^{\lambda(F_k)}\rangle \cong (\mathbb{Z}_2)^k & \text{if } x \in \text{int}(F_1 \cap \cdots \cap F_k), \end{array}$$

where $\langle(\textbf{-1})^{\lambda(F_1)}, \ldots, (\textbf{-1})^{\lambda(F_k)}\rangle \subset (\mathbb{Z}_2)^n$ denotes the subgroup generated by $(\textbf{-1})^{\lambda(F_i)}$ for $i = 1, \ldots, k$. The small cover $(\mathbb{Z}_2)^n \times P/\sim$ is usually denoted by $M(P, \lambda)$.

Let us look at two examples.

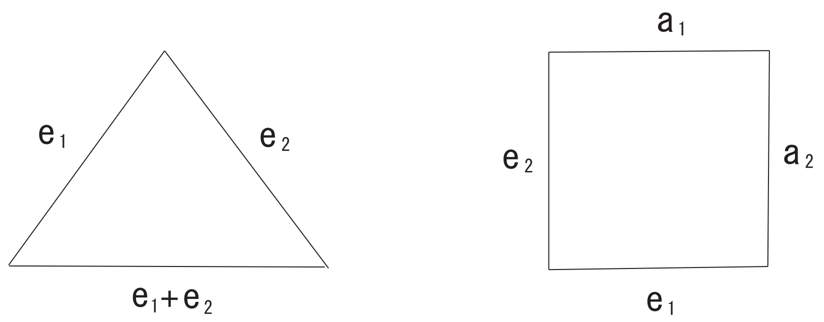

Figure 1: Projective characteristic functions. In the left triangle and the right square, functions satisfy $(\star)$ on each vertex, where $e_1,\ e_2$ are the canonical basis vectors of $(\mathbb{Z}/2\mathbb{Z})^2$ and $a_1,\ a_2 \in (\mathbb{Z}/2\mathbb{Z})^2$.

As is well known, in Figure 1, the left example corresponds to $\mathbb{R}P^2$ with $(\mathbb{Z}_2)^2$-action as in Example 2.1 and the right example corresponds to the following two small covers: (1) if $a_1 = e_1$ and $a_2 = e_2$ then the small cover is $S^1 \times S^1 (\simeq T^2)$ with the diagonal $\mathbb{Z}_2 \times \mathbb{Z}_2$-action (see Example 2.3); (2) otherwise, the small cover is $\mathcal{K}^2$ in Example 2.2.

Summing up, we have the following relationship:

Small covers with $(\mathbb{Z}_2)^n$-actions	$\longrightarrow$ $\longleftarrow$	Simple, convex polytopes with characteristic functions

2.3 Equivariant cohomology and ordinary cohomology of small covers

In this subsection, we recall the equivariant cohomologies and ordinary cohomologies of small covers (see [2, 4] for details). Let $M = M(P, \lambda)$ be an n-dimensional small cover. We denote the facet of P by $\mathcal{F} = \{F_1, \ldots, F_m\}$ such that $\cap_{i=1}^n F_i \neq \emptyset$. Then, we may take the characteristic functions on $F_1, \ldots, F_n$ as

$$\lambda(F_i) = e_i,$$

where $e_1, \ldots, e_n$ are the standard basis vectors of $(\mathbb{Z}/2\mathbb{Z})^n$. That is, we can write the characteristic matrix as

$$\Lambda = (I_n \mid \Lambda'),$$

where I_n is the $(n \times n)$-identity matrix and Λ' is an $(\ell \times n)$-matrix, where $\ell = m - n$.

The *equivariant cohomology* of a G-manifold X is defined by the ordinary cohomology of $EG \times_G X$, where EG is the total space of a universal G-bundle, and denoted by $H_G^*(X)$. In this paper, we assume the coefficient group of cohomology is $\mathbb{Z}/2\mathbb{Z}$. Due to [4], the ring structure of the equivariant cohomology of a small cover M is given by the following formula:

$$H^*_{(\mathbb{Z}_2)^n}(M) \cong \mathbb{Z}/2\mathbb{Z}[\tau_1, \ldots, \tau_m]/\mathcal{I},$$

where the symbol $\mathbb{Z}/2\mathbb{Z}[\tau_1, \ldots, \tau_m]$ represents the polynomial ring generated by the degree 1 elements τ_i ($i = 1, \ldots, m$), and the ideal $\mathcal{I}$ is generated by the following monomial elements:

$$\prod_{i \in I} \tau_i,$$

where I varies over every subset of $\{1, \ldots, m\}$ such that $\cap_{i \in I} F_i = \emptyset$. On the other hand, the ordinary cohomology ring of M is given by

$$H^*(M) \cong H^*_{(\mathbb{Z}_2)^n}(M)/\mathcal{J},$$

where the ideal $\mathcal{J}$ is generated by the following degree 1 homogeneous elements:

$$\tau_i + \lambda_{i1} x_1 + \cdots + \lambda_{i\ell} x_\ell,$$

for $i = 1, \ldots, n$. Here, $(\lambda_{i1} \cdots \lambda_{i\ell})$ is the ith row vector of Λ' ($i = 1, \ldots, n$), and $x_j = \tau_{n+j}$ ($j = 1, \ldots, \ell$).

3 Motivation (Masuda's counterexamples)

In this section, we recall a motivational example, more precisely Masuda's counterexample mentioned in the Introduction.

3.1 Projective bundles associated with vector bundles

Given a k-dimensional real vector bundle ξ over M, we denote by $E(\xi)$ its total space, $\widetilde{\rho} : E(\xi) \to M$ the vector bundle projection, and $F_x(\xi)$ the fibre over $x \in M$, thus $\widetilde{\rho}^{-1}(x) = F_x(\xi)$. Then it is known that the space $P(\xi)$ whose points are the 1-dimensional vector subspaces in the fibre $F_x(\xi)$ for all $x \in M$ is the total space of a fibre bundle over M, with fibre homeomorphic to $(k-1)$-dimensional real projective space. We denote the projection of this fibre bundle, known as the *projective bundle of the vector bundle* ξ, by $\rho : P(\xi) \to M$, and its fibre $\rho^{-1}(x)$ is often denoted by $P_x(\xi)$.

3.2 Masuda's example

Let $M(q) = P(q\gamma \oplus (b-q)\varepsilon)$ be the projective bundle of $q\gamma \oplus (b-q)\varepsilon$, where γ is the tautological line bundle, ε is the trivial line bundle over $\mathbb{R}P^a$, and $0 \le q \le b$; here a and b are fixed positive integers.

In [10], Masuda proves the following theorem.

Theorem 3.1 (Masuda)**.** *The following two statements hold:*

(1) $H^*(M(q);\mathbb{Z}/2\mathbb{Z}) \cong H^*(M(q');\mathbb{Z}/2\mathbb{Z})$ *if and only if* $q' \equiv q$ *or* $b-q \pmod{2^{h(a)}}$*;*

(2) $M(q) \simeq M(q')$ *if and only if* $q' \equiv q$ *or* $b-q \pmod{2^{k(a)}}$*,*

where $h(a) = \min\{n \in \mathbb{N} \cup \{0\} \mid 2^n \ge a\}$ *and* $k(a) = \#\{n \in \mathbb{N} \mid 0 < n < a \text{ and } n \equiv 0, 1, 2, 4 \pmod 8\}$.

Therefore, in view of Proposition 4.1 below, the following example gives a counterexample to cohomological rigidity of small covers.

Corollary 3.2. *Let* $M(q) = P(q\gamma \oplus (17-q)\varepsilon)$ *be the above projective bundle over* $\mathbb{R}P^{10}$*, i.e.,* $a = 10$ *and* $b = 17$*. Then* $H^*(M(0);\mathbb{Z}/2\mathbb{Z}) \cong H^*(M(1);\mathbb{Z}/2\mathbb{Z})$ *but* $M(0) \not\simeq M(1)$.

Proof. By the definition of $h(a)$ and $k(a)$, we have $h(10) = 4$, $k(10) = 5$. Because $b = 17$, we have $0 \equiv 17 - 1 \pmod{2^{h(10)}}$. Therefore, the cohomology rings of $M(0)$ and $M(1)$ are isomorphic. However, $0 \not\equiv 17 - 1 \pmod{2^{k(10)}}$. It follows that $M(0)$ and $M(1)$ are not homeomorphic. □

4 Projective bundles over small covers

In this section, we introduce some notations and basic results for projective bundles over small covers. We first recall the definition of a G-equivariant vector bundle over a G-space M (also see the notations in Section 3). A *G-equivariant vector bundle* is a vector bundle ξ over the G-space M together with a lift of the G-action to $E(\xi)$ by fibrewise linear transformations, i.e., $E(\xi)$ is also a G-space, the projection $E(\xi) \to M$ is G-equivariant and the induced fibre isomorphism between $F_x(\xi)$ and $F_{gx}(\xi)$ is linear, where $x \in M$ and $g \in G$.

Henceforth, we assume M is an n-dimensional small cover, and ξ is a k-dimensional, $(\mathbb{Z}_2)^n$-equivariant vector bundle over M. One can easily show that the following proposition gives a criterion for the projective bundle $P(\xi)$ to be a small cover.

Proposition 4.1. *The projective bundle $P(\xi)$ of ξ is a small cover if and only if the vector bundle ξ decomposes into the Whitney sum of line bundles, i.e., $\xi \equiv \gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \gamma_k$.*

By using the fact that $P(\xi \otimes \gamma) \simeq P(\xi)$ (homeomorphic) for all line bundles γ (e.g. see [11]) and the above Proposition 4.1, we have the following corollary.

Corollary 4.2. *Let M be a small cover, and ξ be the Whitney sum of some k line bundles over M. Then the small cover $P(\xi)$ is homeomorphic to*

$$P(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \varepsilon),$$

where the vector bundles γ_i ($i = 1, \ldots, k-1$) are line bundles.

In this article, a *projective bundle over a small cover* means the projective bundle in Corollary 4.2 (also see Section 1).

4.1 Structure of projective bundles over small covers

In this section, we exhibit the structure of projective bundles over small covers. First, we recall the moment-angle complex of small covers. Let P be a simple, convex polytope and $\mathcal{F}$ the set of its facets $\{F_1, \dots, F_m\}$. We denote by $\mathcal{Z}_P$ the manifolds

$$\mathcal{Z}_P = (\mathbb{Z}_2)^m \times P/\sim,$$

where $(t,\ p) \sim (t',\ p)$ is defined by $t^{-1}t' \in \prod_{p\in F_i} \mathbb{Z}_2(i)$ ($\mathbb{Z}_2(i) \subset (\mathbb{Z}_2)^m$ is the subgroup generated by the ith factor), and we call it a *moment-angle manifold* of P. We note that if $P = M^n/(\mathbb{Z}_2)^n$ then there is a subgroup $K \subset (\mathbb{Z}_2)^m$ such that $K \cong (\mathbb{Z}_2)^{m-n}$ and K acts freely on $\mathcal{Z}_P$. Therefore, we can denote the small cover $M = \mathcal{Z}_P/(\mathbb{Z}_2)^\ell$ for $\ell = m - n$.

Since $[M; B\mathbb{Z}_2] = H^1(M; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^\ell$ (see [4, 17]), we see that all line bundles γ can be written as follows:

$$\gamma \equiv \mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} \mathbb{R}_\rho,$$

where $(\mathbb{Z}_2)^\ell$ acts on $\mathbb{R}_\rho = \mathbb{R}$ by some representation $\rho : (\mathbb{Z}_2)^\ell \to \mathbb{Z}_2$. Moreover, its total Stiefel-Whitney class is $w(\mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} \mathbb{R}) = 1 + \delta_1 x_1 + \cdots + \delta_\ell x_\ell$, where $(\delta_1, \dots, \delta_\ell) \in (\mathbb{Z}/2\mathbb{Z})^\ell$ is induced by a representation $(\mathbb{Z}_2)^\ell \to \mathbb{Z}_2$, i.e.,

$$(-1, \dots, -1) \mapsto (-1)^{\delta_1} \cdots (-1)^{\delta_\ell},$$

and $x_1,\ \dots,\ x_\ell$ are the degree 1 generators of $H^*(M)$ introduced in Section 2.3. Therefore, by using Corollary 4.2, all projective bundles of small covers are as follows:

$$P(\xi) = \mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} (\mathbb{R}^k - \{0\})/\mathbb{R}^* = \mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} \mathbb{R}P^{k-1}, \tag{4.1}$$

where

$$\xi = \mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} \mathbb{R}^k$$

with the $(\mathbb{Z}_2)^\ell$ representation space $\mathbb{R}^k = \mathbb{R}_{\alpha_1} \oplus \cdots \oplus \mathbb{R}_{\alpha_k}$ such that

$$\alpha_i : (\mathbb{Z}_2)^\ell \to \mathbb{Z}_2$$

where $i = 1, \dots, k$ and α_k is the trivial representation. Then we may denote the projective bundle of a small cover by

$$\mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} \mathbb{R}P^{k-1} = P(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \varepsilon),$$

where $\gamma_i = \mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} \mathbb{R}_{\alpha_i}$ $(i = 1, \cdots, k-1)$ satisfies $w(\gamma_i) = 1 + \delta_{1i}x_1 + \cdots + \delta_{\ell i}x_\ell$ for $(\delta_{1i}, \cdots, \delta_{\ell i}) \in (\mathbb{Z}/2\mathbb{Z})^\ell$, which is induced by the representation $\alpha_i : (\mathbb{Z}_2)^\ell \to \mathbb{Z}_2$.

Let $(I_n \mid \Lambda) \in M(m, n; \mathbb{Z}/2\mathbb{Z})$ be the characteristic matrix of M. Using the above construction of projective bundles and computing their characteristic functions, we have the following proposition.

Proposition 4.3. *Let $P(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \varepsilon)$ be the projective bundle over M. Then its orbit polytope is $P^n \times \Delta^{k-1}$, and its characteristic matrix is as follows:*

$$\begin{pmatrix} I_n & O & \Lambda & \boldsymbol{0} \\ O & I_{k-1} & \Lambda_\xi & \boldsymbol{1} \end{pmatrix}, \tag{4.2}$$

where $P^n = M/(\mathbb{Z}_2)^n$ and

$$\Lambda_\xi = \begin{pmatrix} \delta_{11} & \cdots & \delta_{\ell 1} \\ \vdots & \ddots & \vdots \\ \delta_{1,k-1} & \cdots & \delta_{\ell,k-1} \end{pmatrix}.$$

4.2 New characteristic function of projective bundles over 2-dimensional small covers

In order to illustrate the main result, we introduce a new characteristic function (matrix). As an easy example, we only consider the case when $n = 2$. Let $(I_2 \mid \Lambda)$ be the characteristic matrix of M^2, where $\Lambda \in M(2, \ell; \mathbb{Z}/2\mathbb{Z})$ for $\ell = m - 2$ (m is the number of facets of $P^2 = M/(\mathbb{Z}_2)^2$). By using Proposition 4.3, the characteristic matrix of $P(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \varepsilon)$ is

$$\begin{pmatrix} I_2 & O & \Lambda & \mathbf{0} \\ O & I_{k-1} & \Lambda_\xi & \mathbf{1} \end{pmatrix}, \tag{4.3}$$

where

$$\begin{pmatrix} \Lambda \\ \Lambda_\xi \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{\ell 1} \\ \lambda_{12} & \cdots & \lambda_{\ell 2} \\ \delta_{11} & \cdots & \delta_{\ell 1} \\ \vdots & \ddots & \vdots \\ \delta_{1,k-1} & \cdots & \delta_{\ell,k-1} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_\ell \\ \mathbf{b}_1 & \cdots & \mathbf{b}_\ell \end{pmatrix},$$

where $\mathbf{a}_i \in (\mathbb{Z}/2\mathbb{Z})^2$ and $\mathbf{b}_i \in (\mathbb{Z}/2\mathbb{Z})^{k-1}$ for $i = 1, \ldots, \ell$. Therefore, in order to determine the projective bundles over M^2, it is sufficient to consider the following characteristic function: for an m-gon P^2 and its facets $\mathcal{F}$, the function

$$\lambda_P : \mathcal{F} = \{F_1, \ldots, F_m\} \to (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^{k-1}$$

satisfies $\lambda_P(F_1) = e_1 \times \mathbf{0}$, $\lambda_P(F_2) = e_2 \times \mathbf{0}$ (denote them simply $e_1 = (1,\ 0)$, $e_2 = (0,\ 1)$, respectively) and

$$\det(\lambda_P(F_i)\ \lambda_P(F_j)\ X_1\ \cdots\ X_{k-1}) = 1 \tag{4.4}$$

for $F_i \cap F_j \neq \emptyset$ $(i \neq j)$ and all $\{X_1, \ldots, X_{k-1}\} \subset \{\mathbf{0} \times e'_1, \ldots, \mathbf{0} \times e'_{k-1},\ \mathbf{0} \times \mathbf{1}\}$, where e'_i $(i = 1, \ldots, k-1)$ is the canonical basis in $(\mathbb{Z}/2\mathbb{Z})^{k-1}$. We call this function a *projective characteristic function* (we can easily generalize this notion to the general dimension n, not only 2), and $(P^2,\ \lambda_P)$ corresponds to the projective bundle over the 2-dimensional small cover (over P^2). Figure 2 is an illustration of projective characteristic functions.

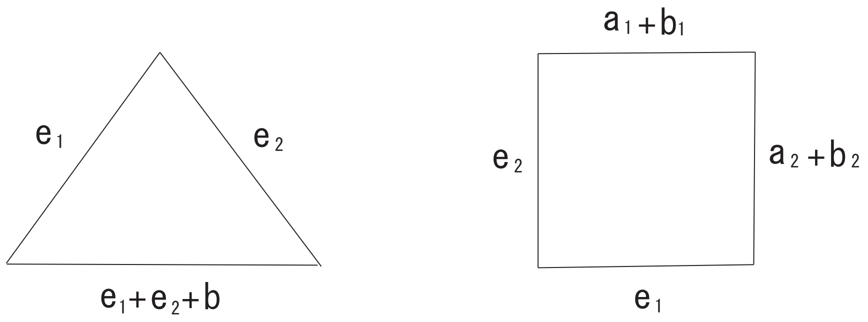

Figure 2: Projective characteristic functions. In the left triangle and the right square, functions satisfy (4.4) on each vertex, where $a_1,\ a_2 \in (\mathbb{Z}/2\mathbb{Z})^2$.

Note that in Figure 2, if we put $b = 0$ and $b_1 = b_2 = 0 \in (\mathbb{Z}/2\mathbb{Z})^{k-1}$, then this gives ordinary characteristic functions on the triangle and the square. Therefore, we can regard such a forgetful map of the $(\mathbb{Z}/2\mathbb{Z})^{k-1}$ part, $f : (P^2,\ \lambda_P) \to (P^2,\ \lambda)$, as the equivariant projection $P(\xi) \to M^2$.

5 Basic 2-dimensional small covers and classification of their projective bundles

In order to state the main result, in this section, we introduce a construction theorem for 2-dimensional small covers and two classification results for projective bundles.

5.1 Construction theorem for 2-dimensional small covers

First, we introduce Proposition 5.1. Note that a 2-dimensional small cover can be induced by a 2-dimensional polytope and characteristic functions such as those in Figure 3, where $e_1 = (1,\ 0)$, $e_2 = (0,\ 1)$, $e_1 + e_2 = (1,\ 1)$ and the *n-gon* is the 2-dimensional polytope with n facets (or n vertices).

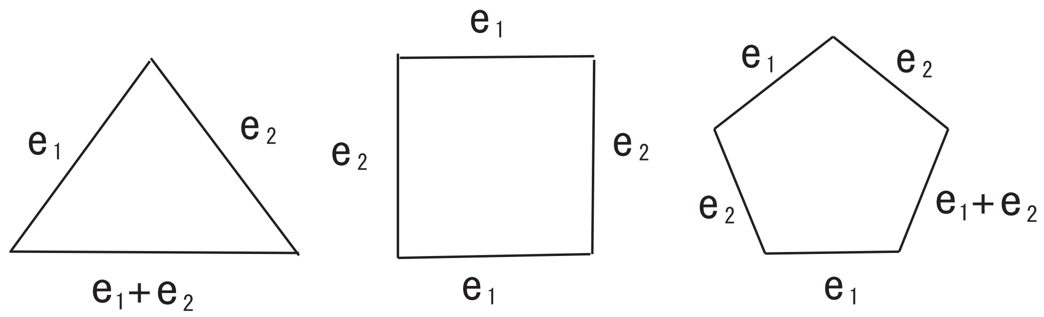

Figure 3: The left figure is $(\Delta^2,\ \lambda_0)$ and the middle figure is $(I^2,\ \lambda_0^I)$. We see that $M(\Delta^2,\ \lambda_0) = \mathbb{R}P^2$ and $M(I^2,\ \lambda_0^I) = T^2$, where $\mathbb{R}P^2$ and T^2 have the standard $(\mathbb{Z}_2)^2$-actions. The third figure is the 5-gon with characteristic functions.

We have the following construction theorem for 2-dimensional small covers based on an argument from [16].

Proposition 5.1 (Construction theorem)**.** *Let M^2 be a 2-dimensional small cover. Then M^2 is equivariantly homeomorphic to an equivariant connected sum of $\mathbb{R}P^2$ and T^2 with standard $(\mathbb{Z}_2)^2$-actions.*

Therefore, the real projective space $\mathbb{R}P^2$ and torus T^2 with standard $(\mathbb{Z}_2)^2$-actions are the basic small covers in the realm of 2-dimensional small covers.

5.2 Topological classification of projective bundles over $\mathbb{R}P^2$ and T^2

Next, we classify the topological types of projective bundles over basic small covers, i.e., $\mathbb{R}P^2$ and T^2.

The classification of projective bundles over $\mathbb{R}P^2$ is known by Masuda's paper [10]. Thanks to [10], we have $q \equiv q'$ or $k-q'$ (mod 4) if and only if $S^2 \times_{\mathbb{Z}_2} P(q\gamma \oplus (k-q)\varepsilon) \simeq$

$S^2 \times_{\mathbb{Z}_2} P(q'\gamma \oplus (k-q')\varepsilon)$. Note that a line bundle over $\mathbb{R}P^2$ is either the tautological line bundle γ or the trivial line bundle ε. Therefore, by Proposition 4.1, we see that the projective bundles over $\mathbb{R}P^2$ are only these types. Hence, we can easily obtain the following proposition.

Proposition 5.2. *Let $P(q\gamma \oplus (k-q)\varepsilon)$ be a projective bundle over $\mathbb{R}P^2$. Then its topological type is one of the following.*

(1) *The case $k \equiv 0 \pmod 4$:*

   (a) *if $q \equiv 0 \pmod 4$, then $P(q\gamma \oplus (k-q)\varepsilon) \simeq \mathbb{R}P^2 \times \mathbb{R}P^{k-1}$;*

   (b) *if $q \equiv 1,\ 3 \pmod 4$, then $P(q\gamma \oplus (k-q)\varepsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(\gamma \oplus (k-1)\varepsilon)$;*

   (c) *if $q \equiv 2 \pmod 4$, then $P(q\gamma \oplus (k-q)\varepsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(2\gamma \oplus (k-2)\varepsilon)$.*

(2) *The case $k \equiv 1 \pmod 4$:*

   (a) *if $q \equiv 0,\ 1 \pmod 4$, then $P(q\gamma \oplus (k-q)\varepsilon) \simeq \mathbb{R}P^2 \times \mathbb{R}P^{k-1}$;*

   (b) *if $q \equiv 2,\ 3 \pmod 4$, then $P(q\gamma \oplus (k-q)\varepsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(2\gamma \oplus (k-2)\varepsilon)$.*

(3) *The case $k \equiv 2 \pmod 4$:*

   (a) *if $q \equiv 0,\ 2 \pmod 4$, then $P(q\gamma \oplus (k-q)\varepsilon) \simeq \mathbb{R}P^2 \times \mathbb{R}P^{k-1}$;*

   (b) *if $q \equiv 1 \pmod 4$, then $P(q\gamma \oplus (k-q)\varepsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(\gamma \oplus (k-1)\varepsilon)$;*

   (c) *if $q \equiv 3 \pmod 4$, then $P(q\gamma \oplus (k-q)\varepsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(3\gamma \oplus (k-3)\varepsilon)$.*

(4) *The case $k \equiv 3 \pmod 4$:*

   (a) *if $q \equiv 0,\ 3 \pmod 4$, then $P(q\gamma \oplus (k-q)\varepsilon) \simeq \mathbb{R}P^2 \times \mathbb{R}P^{k-1}$;*

   (b) *if $q \equiv 1,\ 2 \pmod 4$, then $P(q\gamma \oplus (k-q)\varepsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(\gamma \oplus (k-1)\varepsilon)$.*

Note that the moment-angle manifold over $\mathbb{R}P^2$ is S^2.

Finally, we classify projective bundles over T^2. Let γ_i be the pull back of the canonical line bundle over S^1 by the ith factor projection $\pi_i : T^2 \to S^1$ $(i = 1,\ 2)$. We can easily show that line bundles over T^2 are completely determined by their first Stiefel-Whitney classes via $[T^2,\ B\mathbb{Z}_2] \cong H^1(T^2;\ \mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$. Therefore, all of the line bundles over T^2 are ε, γ_1, γ_2 and $\gamma_1 \otimes \gamma_2$. By the definition of γ_i, we can easily show that $\gamma_i \oplus \gamma_i = \pi_i^*(\gamma \oplus \gamma) = \pi_i^*(2\varepsilon) = 2\varepsilon$. Therefore, we also have

$$(\gamma_1 \otimes \gamma_2) \oplus (\gamma_1 \otimes \gamma_2) = \gamma_1 \otimes (\gamma_2 \oplus \gamma_2) = \gamma_1 \otimes 2\varepsilon = \gamma_1 \oplus \gamma_1 = 2\varepsilon.$$

Hence, we have the following proposition.

Proposition 5.3. *Let $P(\xi)$ be a projective bundle over T^2. Then its topological type is one of the following.*

(1) *The case $k \equiv 0 \pmod 2$:*

 (a) $P(k\varepsilon) \simeq T^2 \times \mathbb{R}P^{k-1}$;

 (b) $P((\gamma_1 \otimes \gamma_2) \oplus (k-1)\varepsilon) \simeq P(\gamma_1 \oplus \gamma_2 \oplus (k-2)\varepsilon) \simeq T^2 \times_{(\mathbb{Z}_2)^2} P(\mathbb{R}_{\rho_1} \oplus \mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-2})$;

 (c) $P(\gamma_1 \oplus (k-1)\varepsilon) \simeq P((\gamma_1 \otimes \gamma_2) \oplus \gamma_2 \oplus (k-2)\varepsilon) \simeq T^2 \times_{(\mathbb{Z}_2)^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}}^{k-1})$;

 (d) $P(\gamma_2 \oplus (k-1)\varepsilon) \simeq P((\gamma_1 \otimes \gamma_2) \oplus \gamma_1 \oplus (k-2)\varepsilon) \simeq T^2 \times_{(\mathbb{Z}_2)^2} P(\mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-1})$;

(2) *The case $k \equiv 1 \pmod 2$:*

 (a) $P(k\varepsilon) \simeq T^2 \times \mathbb{R}P^{k-1}$;

 (b) $P((\gamma_1 \otimes \gamma_2) \oplus (k-1)\varepsilon) \simeq P(\gamma_1 \oplus (k-1)\varepsilon) \simeq P(\gamma_2 \oplus (k-1)\varepsilon) \simeq T^2 \times_{(\mathbb{Z}_2)^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}}^{k-1})$;

 (c) $P((\gamma_1 \oplus \gamma_2) \oplus (k-2)\varepsilon) \simeq P((\gamma_1 \otimes \gamma_2) \oplus \gamma_1 \oplus (k-2)\varepsilon) \simeq P((\gamma_1 \otimes \gamma_2) \oplus \gamma_2 \oplus (k-2)\varepsilon) \simeq T^2 \times_{(\mathbb{Z}_2)^2} P(\mathbb{R}_{\rho_1} \oplus \mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-2})$,

where $\rho_i : (\mathbb{Z}_2)^2 \to \mathbb{Z}_2$ is the ith projection and $\underline{\mathbb{R}}$ is the trivial representation space.

Note that the moment-angle manifold over T^2 is T^2 itself.

6 Main Theorem

In this section, we state our main theorem. Before doing so, we introduce a new operation.

For two polytopes with projective characteristic functions, we can do the connected sum operation which is compatible with projective characteristic functions as indicated in Figure 4. Then we get a new polytope with the projective characteristic

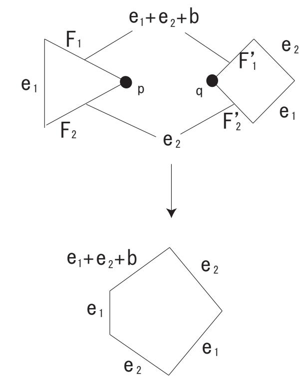

Figure 4: We can construct a connected sum at two vertices with the same projective characteristic function.

function. We call this operation a *projective connected sum* and denote it by $\sharp^{\Delta^{k-1}}$.

More precisely, this operation is defined as follows. Let p and q be vertices in 2-dimensional polytopes with projective characteristic functions (P, λ_P) and $(P', \lambda_{P'})$, respectively. Here, we assume that the target spaces of λ_P and $\lambda_{P'}$ are the same $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^{k-1}$, i.e., the corresponding projective bundles have the same fibre $\mathbb{R}P^{k-1}$. Moreover, we assume that $\lambda_P(F_1) = \lambda_{P'}(F_1')$ and $\lambda_P(F_2) = \lambda_{P'}(F_2')$ for facets $\{F_1, F_2\}$ around p and $\{F_1', F_2'\}$ around q, i.e., $F_1 \cap F_2 = \{p\}$ and $F_1' \cap F_2' = \{q\}$. Then we can do the connected sum of two polytopes P and P' at these vertices by gluing each pair of facets F_i and F_i'. Thus, we get a polytope with projective characteristic functions $(P\sharp^{\Delta^{k-1}}P', \ \lambda_{P\sharp^{\Delta^{k-1}}P'})$ from (P, λ_P) and $(P', \lambda_{P'})$ (also see Figure 4).

Note that, from the geometric point of view, the inverse image of vertices of polytopes with projective characteristic functions corresponds to the projective space $\mathbb{R}P^{k-1}$. Therefore, a geometric interpretation of this operation is an equivariant gluing along the fibre $\mathbb{R}P^{k-1}$, i.e., fibre sum of two fibre bundles.

Now we may state the main theorem of this article.

Theorem 6.1. *Let $P(\xi)$ be a projective bundle over a 2-dimensional small cover M^2. Then $P(\xi)$ can be constructed from one of the projective bundles $P(\kappa)$ over $\mathbb{R}P^2$ in Proposition 5.2 and $P(\zeta)$ over T^2 in Proposition 5.3 by using the projective connected sum $\sharp^{\Delta^{k-1}}$.*

If $k = 1$, then the projective characteristic function is the ordinary characteristic function, i.e., the fibre dimension is 0. Therefore, we may regard this theorem as a generalization of the construction theorem of 2-dimensional small covers (Proposition 5.1).

7 Appendix: Topological classification of projective bundles over the 1-dimensional small cover

In closing this paper, we explain a topological classification of projective bundles over the 1-dimensional small cover.

It is easy to show that the 1-dimensional small cover is the 1-dimensional circle S^1 and its moment-angle manifold is S^1 itself, i.e., the moment-angle manifold $S^1 \subset \mathbb{R}^2$ has the diagonal free $\mathbb{Z}_2$-action and its quotient is the 1-dimensional small cover S^1. Therefore, the classification problem corresponds to the classification of the following projective bundles of $S^1 \times_{\mathbb{Z}_2} (q\gamma \oplus (k-q)\varepsilon)$:

$$S^1 \times_{\mathbb{Z}_2} P(q\gamma \oplus (k-q)\varepsilon),$$

where γ is the non-trivial line bundle and ε is the trivial line bundle over S^1 and $0 \le q \le k-1$ (also see [10]).

Because $[S^1,\ BO(k)] = \mathbb{Z}_2$, the vector bundles over S^1 can be classified by their first Stiefel-Whitney classes. Therefore, we see that

(1) if $q \equiv 0 \pmod 2$, then $S^1 \times_{\mathbb{Z}_2} (q\gamma \oplus (k-q)\varepsilon) \equiv S^1 \times_{\mathbb{Z}_2} k\varepsilon \simeq S^1 \times \mathbb{R}^k$;

(2) if $q \equiv 1 \pmod 2$, then $S^1 \times_{\mathbb{Z}_2} (q\gamma \oplus (k-q)\varepsilon) \equiv S^1 \times_{\mathbb{Z}_2} (\gamma \oplus (k-1)\varepsilon) \simeq (S^1 \times_{\mathbb{Z}_2} \mathbb{R}) \times \mathbb{R}^{k-1}$.

Because of the following three well-known facts: $P(\xi \otimes \gamma) \simeq P(\xi)$; $H^*(S^1 \times_{\mathbb{Z}_2} P(\xi);\ \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x,\ y]/\langle x^2,\ y^k + y^{k-1}w_1(\pi^*\xi)\rangle$ for $\deg x = \deg y = 1$ (by the Borel-Hirzebruch formula, see [3]); and the injectivity of the induced homomorphism $\pi^* : H^*(S^1;\ \mathbb{Z}/2\mathbb{Z}) \to H^*(P(\xi);\ \mathbb{Z}/2\mathbb{Z})$ of $\pi : P(\xi) \to S^1$ (see [14]), we have the following proposition.

Proposition 7.1. *Let $P(q\gamma \oplus (k-q)\varepsilon)$ be a projective bundle over S^1. Then its topological type is one of the following:*

(1) *if $k \equiv 1 \pmod 2$ or $q \equiv 0 \pmod 2$, then $P(q\gamma \oplus (k-q)\varepsilon) = S^1 \times \mathbb{R}P^{k-1}$;*

(2) *otherwise, i.e., if* $k \equiv 0 \pmod 2$ *and* $q \equiv 1 \pmod 2$*, then* $P(q\gamma \oplus (k-q)\varepsilon) = S^1 \times_{\mathbb{Z}_2} \mathbb{R}P^{k-1}$,

where $\mathbb{Z}_2$ *acts freely on* S^1 *and on the first coordinate of* $\mathbb{R}P^{k-1}$.

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The primary Smith sets of finite Oliver groups

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ABSTRACT. Let G be a finite group. There is an equivalence relation of finite dimensional real G-modules (i.e. real G-representation spaces) called *the Smith equivalence*. The *Smith set* of G is the set of all differences of Smith equivalent real G-modules in the real representation ring of G. A part of the Smith set of G is called the *primary Smith set* of G. In this paper we discuss the primary Smith sets of Oliver groups G.

1 Results

Throughout this paper let G be a finite group. Let $\mathfrak{X}$ be a family of smooth G-manifolds. Real G-modules (i.e. real G-representation spaces) V and W (of finite dimension) are called $\mathfrak{X}$*-related* and written $V \sim_{\mathfrak{X}} W$ if there exists $X \in \mathfrak{X}$ such that V and W are G-isomorphic to the tangential representations $T_x(X)$ and $T_y(X)$, respectively, for some x and y in the G-fixed point set X^G. Let $\mathrm{RO}(G)$ denote the real representation ring of G and let $\mathrm{RO}(G, \mathfrak{X})$ denote the subset of $\mathrm{RO}(G)$ consisting of all $[V] - [W]$ such that $V \sim_{\mathfrak{X}} W$. Let $\mathfrak{S}$ (resp. $\mathfrak{S}_{\mathrm{ht}}$) denote the family of all standard spheres (resp. homotopy spheres) equipped with smooth G-actions with exactly two G-fixed points. Since the connected sum of two homotopy spheres of the same dimension is a homotopy sphere, $\mathfrak{S}_{\mathrm{ht}}$-relation is an equivalence relation. Real G-modules V and W are called *Smith equivalent* if V and W are $\mathfrak{S}_{\mathrm{ht}}$-related. The set $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})$ is called the *Smith set* of G.

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Let $\mathcal{S}(G)$ denote the set of all subgroups of G. For $\mathcal{A} \subset \mathrm{RO}(G)$ and $\mathcal{F}, \mathcal{G} \subset \mathcal{S}(G)$, let

$$\mathcal{A}^{\mathcal{F}} = \{[V] - [W] \in \mathcal{A} \mid V^H = 0 = W^H \quad (\forall H \in \mathcal{F})\},$$
$$\mathcal{A}_{\mathcal{G}} = \{[V] - [W] \in \mathcal{A} \mid \mathrm{res}^G_H V \cong_H \mathrm{res}^G_H W \quad (\forall H \in \mathcal{G})\},$$
$$\mathcal{A}^{\mathcal{F}}_{\mathcal{G}} = (\mathcal{A}^{\mathcal{F}})_{\mathcal{G}}.$$

If all minimal elements of $\mathcal{F}$ are normal subgroups of G, then $\mathcal{A}^{\mathcal{F}}_{\mathcal{G}}$ coincides with $\mathcal{A}^{\mathcal{F}} \cap \mathcal{A}_{\mathcal{G}}$. Let $\mathcal{P}(G)$ (resp. $\mathcal{P}(G)_{\mathrm{odd}}$) denote the set of subgroups P of G such that the order of P is a prime power (resp. an odd-prime power). The set $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})_{\mathcal{P}(G)}$ is called the *primary Smith set* of G. With this notation, we can say that C. Sanchez [23] proved the implication $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}}) \subset \mathrm{RO}(G)_{\mathcal{P}(G)_{\mathrm{odd}}}$ and S. Cappell-J. Shaneson showed the fact $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}}) \not\subset \mathrm{RO}(G)_{\mathcal{P}(G)}$ in the case $G = C_8$ the cyclic group of order 8. Actually, $\mathrm{RO}(C_8)_{\mathcal{P}(G)} = 0$ follows from definition while the fact $\mathrm{RO}(C_8, \mathfrak{S}_{\mathrm{ht}}) \neq 0$ is proved in [2].

In the present paper we shall prove the following theorem which shows that the difference of the Smith set and the primary Smith set of G is a finite set.

Theorem 1. *Let G be a finite group and let $\mathfrak{X}$ be either $\mathfrak{S}$ or $\mathfrak{S}_{\mathrm{ht}}$. Then the set $\mathrm{RO}(G, \mathfrak{X}) \smallsetminus \mathrm{RO}(G, \mathfrak{X})_{\mathcal{P}(G)}$ is empty or consists of finitely many elements.*

In Theorem 1, if we have $\mathrm{RO}(G, \mathfrak{X}) \smallsetminus \mathrm{RO}(G, \mathfrak{X})_{\mathcal{P}(G)} \neq \emptyset$, then both $\mathrm{RO}(G, \mathfrak{X})$ and $\mathrm{RO}(G, \mathfrak{X})_{\mathcal{P}(G)}$ cannot be subgroups of $\mathrm{RO}(G)$ because as a group, $\mathrm{RO}(G)$ is a finitely generated free abelian group. If $\mathrm{RO}(G, \mathfrak{X})$ is an infinite set, then $\mathrm{RO}(G, \mathfrak{X})_{\mathcal{P}(G)}$ is an infinite set and hence we may say that $\mathrm{RO}(G, \mathfrak{X})_{\mathcal{P}(G)}$ is a 'main part' of $\mathrm{RO}(G, \mathfrak{X})$ on account of cardinality, although one may be interested in the 'singular finite set' $\mathrm{RO}(G, \mathfrak{X}) \smallsetminus \mathrm{RO}(G, \mathfrak{X})_{\mathcal{P}(G)}$ more than the infinite set $\mathrm{RO}(G, \mathfrak{X})_{\mathcal{P}(G)}$.

In 1996, E. Laitinen posed a conjecture concerned with the Smith set and this conjecture motivated us to compute $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})_{\mathcal{P}(G)}$ for various groups G. We recall here that a finite group G is called an *Oliver group* if G can act smoothly on a disk without G-fixed points, cf. [16]. In other words, G is an Oliver group if G does not admit a normal series $P \trianglelefteq H \trianglelefteq G$ such that P and G/H are of prime power order and H/P is cyclic. The conjecture posed by E. Laitinen appeared in print for the first time in [6, Appendix], and it can be restated as follows.

Conjecture (E. Laitinen**).** Let G be an Oliver group. If $\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} \neq 0$, then $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})_{\mathcal{P}(G)} \neq 0$.

For a prime p, let $\mathcal{N}_p(G)$ denote the set of normal subgroups N of G such that $|G/N| = 1$ or p. The following theorem goes back to [8] and [3].

Theorem 2. *For an arbitrary finite group G, the set* $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})$ *is contained in* $\mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)_{\mathrm{odd}}}$. *If a Sylow 2-subgroup of G is normal in G, then* $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})$ *is contained in the set*

$$\bigcap_{p:\ prime \geq 5} \ \bigcap_{N \in N_p(G)} \left(\mathrm{RO}(G)^{\mathcal{N}_2(G) \cup \mathcal{N}_3(G) \cup \{N\}}_{\mathcal{P}(G)_{\mathrm{odd}}} \cup \mathrm{RO}(G)^{\mathcal{N}_2(G) \cup \mathcal{N}_3(G)}_{\mathcal{P}(G)_{\mathrm{odd}} \cup \{N\}} \right).$$

Using this theorem, one can give various counterexamples to Laitinen's Conjecture, as was done for example in [8], [3], [18], [26].

For a prime p, let $G^{\{p\}}$ denote the smallest normal subgroup H of G such that $|G/H|$ is a power of p. Let $\mathcal{L}(G)$ denote the set of all subgroups H of G containing $G^{\{p\}}$ for some prime p. A real G-module V is called $\mathcal{L}(G)$*-free* if $V^{G^{\{p\}}} = 0$ for all primes p. Let G^{nil} denote the smallest normal subgroup H of G such that G/H is nilpotent. Then we have

$$G^{\mathrm{nil}} = \bigcap_{p \mid |G|} G^{\{p\}}.$$

A real G-module V is called a *gap module* if V is $\mathcal{L}(G)$-free and

$$\dim V^P > 2 \dim V^H$$

for all $P \in \mathcal{P}(G)$ and $H \in \mathcal{S}(G)$ with $P \subsetneq H$. A finite group G is called a *gap group* if there exists a gap real G-module. An Oliver group G is a gap group if any one of the following conditions is satisfied.

(1) $G = G^{\{2\}}$ (cf. [4, Theorem 2.3]).

(2) $G \neq G^{\{p\}}$ holds for at least two odd primes p (cf. [4, Theorem 2.3]).

(3) A Sylow 2-subgroup of G is a normal subgroup of G (cf. [14, Proposition 4.3]).

Further information on gap groups can be found in [25].

The implication

$$\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset \mathrm{RO}(G, \mathfrak{S}) \tag{1.1}$$

has been proved for various Oliver groups G. Its version for abelian groups G of odd order is due to T. Petrie [19]–[22], that for perfect groups G is due to E. Laitinen-K. Pawałowski [6], and that for gap Oliver groups G is due to K. Pawałowski-R. Solomon [17]. It yields the following theorem.

Theorem 3 (K. Pawałowski-R. Solomon**).** *If G is a gap Oliver group, then the equalities*

$$\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S})^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})^{\mathcal{L}(G)}_{\mathcal{P}(G)}$$

hold.

A key procedure to prove the implication (1.1) is to delete G-fixed points from spheres (or disks) by using any one of the deleting-inserting theorems in M. Morimoto-K. Pawałowski [13] (M. Morimoto-E. Laitinen-K. Pawałowski [5] for perfect G) and M. Morimoto [9]. Since G is a perfect group if and only if $G = G^{\mathrm{nil}}$, Theorem 3 implies the following result that was first obtained by E. Laitinen-K. Pawałowski [6].

Corollary (E. Laitinen-K. Pawałowski**).** *If G is a perfect group, then*

$$\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})_{\mathcal{P}(G)}.$$

The following corollary is also immediately obtained from Theorems 2 and 3.

Corollary 4. *If G is a gap Oliver group such that $G/G^{\mathrm{nil}} \cong C_2$, then*

$$\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})_{\mathcal{P}(G)}.$$

Before stating our next result, we recall the fact that an Oliver group G with $G/G^{\mathrm{nil}} \cong C_3$ is a gap group.

Theorem 5. *If G is an Oliver group such that $G/G^{\mathrm{nil}} \cong C_3$, then*

$$\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)} = \begin{cases} \mathrm{RO}(G)^{\{G,\ G^{\{3\}}\}}_{\mathcal{P}(G)} & (G_2 \lhd G) \\ \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} & (G_2 \ntriangleleft G) \end{cases}$$

where G_2 is a Sylow 2-subgroup of G.

For a prime p, a pair (V, W) of real G-modules V and W is called *p-matched* if $\mathrm{res}^G_P V$ is isomorphic to $\mathrm{res}^G_P W$ for a Sylow p-subgroup P of G. The pair is called *of type* 1 if $\dim V^G = 1$ and $W^G = 0$. The pair (V, W) is called *$\mathcal{P}$-matched* if (V, W) is p-matched for any prime p. A 2-matched pair (V, W) is called *special* if $\dim V^G = \dim V^{G^{\{p\}}} = 1$ for any prime $p \neq 3$, $\dim V^{G^{\{3\}}} = 3$, and $W^{G^{\{p\}}} = 0$ for all primes p. Our final theorem reads as follows.

Theorem 6. *Let G be a gap Oliver group such that $G/G^{\mathrm{nil}} \cong C_6$.*

(1) *If G has a normal Sylow 2-subgroup, then*

$$\mathrm{RO}(G)^{\{G,\ G^{\{2\}},\ G^{\{3\}}\}}_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})_{\mathcal{P}(G)}.$$

(2) *If G possesses a special 2-matched pair of real G-modules, then*

$$\mathrm{RO}(G)^{\{G,\ G^{\{2\}}\}}_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})_{\mathcal{P}(G)}.$$

The rest of this paper is organized as follows. In Section 2, by using the work of G. Bredon [1], we prove Theorem 1. In Section 3, we describe key lemmas which we use in the proofs of Theorems 5 and 6.

2 Proof of Theorem 1

For the proof, we first recall G. Bredon's result on smooth C_n-actions on simply connected manifolds, where C_n denotes the cyclic group of order n.

Theorem (G. Bredon [1, Theorem II]**).** *Let M be a simply connected manifold with smooth C_n-action, where $n = p^\alpha$ (p prime, $\alpha \in \mathbb{N}$) and let x and y be C_n-fixed points of M. If $H^i(C_n; \pi_i(M)) = 0$ for $1 \le i \le 2k-2$, then $[T_x(M)] - [T_y(M)]$ is divisible by p^h in $\mathrm{RO}(C_n)$, where*

$$h = \left[\frac{pk-n}{pn-n}\right].$$

The next corollary immediately follows.

Corollary 2.1. *If Σ is a homotopy sphere of dimension d with smooth C_n-action, where $n = 2^\alpha$ ($\alpha \in \mathbb{N}$) and if x and y are C_n-fixed points of Σ, then $T_x(\Sigma) - T_y(\Sigma)$ is divisible by $2^{h'}$ in $\mathrm{RO}(C_n)$, where*

$$h' = \left[\frac{d-n}{n}\right].$$

Proof. First consider the case $d = 1$. Then nontrivial, effective smooth actions on $\Sigma = S^1$ with non-empty C_n-fixed point set are unique up to C_n-diffeomorphism: namely, we have $n = 2$ and the actions are C_n-diffeomorphic to the linear C_2-action on S^1

such that the fixed point set consists of two points. Thus, we see $[T_x(\Sigma)] - [T_y(\Sigma)] = 0$ in $\mathrm{RO}(C_n)$.

Next consider the case $d \geq 2$. Since Σ is $(d-1)$-connected, we would like to find the maximal integer k such that $2k - 2 \leq d - 1$. Clearly we obtain $k = [(d+1)/2]$. Since $d/2 \leq [(d+1)/2]$, Bredon's Theorem implies the conclusion of Corollary 2.1. □

Corollary 2.2 (cf. [21, Ch. 4, Corollary 0.3])**.** *Let Σ be a homotopy sphere of dimension d with smooth C_n-action, where $n = 2^\alpha$ $(\alpha \in \mathbb{N})$ and let x and y be C_n-fixed points of Σ. If*

$$d \geq n(2n+1),$$

then the real C_n-module $T_x(\Sigma)$ is isomorphic to $T_y(\Sigma)$.

Proof. Clearly we may suppose $n \geq 2$. Write $d = an + r$ using non-negative integers a and r such that $0 \leq r < n$. Since $a = (d-r)/n > (d-n)/n$, we have

$$a > 2n \ (\geq 4). \tag{2.1}$$

Suppose $T_x(\Sigma) \ncong T_y(\Sigma)$. Write

$$T_x(\Sigma) = \bigoplus_{i \in I} V_i^{\oplus a_i} \oplus U,$$
$$T_y(\Sigma) = \bigoplus_{j \in J} V_j^{\oplus a_j} \oplus U,$$

where $I \cap J = \emptyset$, a_k $(k \in I \cup J)$ are positive integers, and V_k $(k \in I \cup J)$ are irreducible real C_n-modules which are nonisomorphic to one another. Then the inequality

$$2^{\left[\frac{d-n}{n}\right]} \leq \min\{a_i \mid i \in I\} \leq d \tag{2.2}$$

holds by Corollary 2.2. Then Inequality (2.2) implies

$$2^{a-1} \leq d = an + r < (a+1)n,$$

and hence

$$2^a < 2(a+1)n. \tag{2.3}$$

By (2.1), we get $2^a < a(a+1)$. But for $a \geq 5$, we have

$$2^a \geq 2 + 2a + a(a-1) = 2 + a^2 + a > a(a+1)$$

which is a contradiction. □

Lemma 2.3. *Let G be a finite group and let N denote the maximal order of elements g in G of 2-power order. Let Σ be a homotopy sphere with smooth G-action and let x and y be G-fixed points of Σ. If*

$$\dim \Sigma \geq N(2N+1),$$

then the real P-module $\mathrm{res}^G_P T_x(\Sigma)$ is isomorphic to $\mathrm{res}^G_P T_y(\Sigma)$ for a Sylow 2-subgroup P of G.

Proof. Set $V = T_x(\Sigma)$ and $W = T_x(\Sigma)$ as real G-modules. For the goal $\mathrm{res}_P V \cong \mathrm{res}_P W$, it suffices to show $\mathrm{res}_{C_n} V \cong \mathrm{res}_{C_n} W$ for all $C_n \subset P$. Thus let $C_n \subset P$. Then

$$\dim \Sigma \geq N(2N+1) \geq n(2n+1).$$

Thus, by Corollary 2.2 we get $\mathrm{res}_{C_n} V \cong \mathrm{res}_{C_n} W$. □

Now we recall Sanchez' result.

Lemma 2.4 (C. Sanchez [23])**.** *Let G be a finite group, and let Σ be an integral homology sphere with smooth G-action. If Σ has exactly two G-fixed points x and y, then the real P-module $\mathrm{res}^G_P T_x(\Sigma)$ is isomorphic to $\mathrm{res}^G_P T_y(\Sigma)$ for all Sylow subgroups $P \subset G$ of odd order.*

Theorem 1 immediately follows from Lemmas 2.3 and 2.4.

3 Key lemmas to proofs of Theorems 5 and 6

In this section we introduce lemmas which play a key role in the proofs of Theorems 5 and 6. First we introduce an algebraic lemma which is used to prove Theorem 5.

Lemma 3.1 (B. Oliver [15])**.** *Let G be a finite group. Then the following three statements* (1)–(3) *are equivalent to one another.*

(1) *A Sylow 2-subgroup of G is not normal in G.*

(2) *G possesses a subquotient H/N, where $H \in \mathcal{S}(G)$, $N \triangleleft H$, such that H/N is a dihedral group of order $2p$ for some odd prime p.*

(3) *There exists a 2-matched pair of type 1 consisting of real G-modules.*

We apply this lemma, not to the ambient group G, but to the group G^{nil}. If a Sylow 2-subgroup of G^{nil} is not normal in G^{nil}, then there exists a 2-matched pair (V_0, W_0) of type 1 consisting of real G^{nil}-modules. The induced pair (V, W), where $V = \mathrm{ind}_{G^{\mathrm{nil}}}^{G} V_0$ and $W = \mathrm{ind}_{G^{\mathrm{nil}}}^{G} W_0$, is a 2-matched pair of real G-modules such that $V^{G^{\mathrm{nil}}} \cong \mathbb{R}[G/G^{\mathrm{nil}}]$ and $W^{G^{\mathrm{nil}}} = 0$. If $G/G^{\mathrm{nil}} \cong C_3$, then this pair (V, W) is a special 2-matched pair.

Let U be a real G-module with a G-invariant inner product. For a subgroup H of G, let U_H be the orthogonal complement of U^H in U. We can regard U_H as a real $N_G(H)$-module. Let $U^{\mathcal{L}(G)}$ denote the G-submodule of U spanned by all elements $x \in U$ such that the isotropy subgroup G_x at x belongs to $\mathcal{L}(G)$. In addition, let $U_{\mathcal{L}(G)}$ denote the orthogonal complement of $U^{\mathcal{L}(G)}$ in U. Thus we have

$$U_{\mathcal{L}(G)} = U_G - \bigoplus_{p||G|} U^{G^{\{p\}}}{}_G.$$

For a real G-module X with a G-invariant inner product, we have the real projective space $P(X)$ associated with X and the canonical line bundle γ_Y over $Y = P(X)$. The manifold Y and the total space of γ_Y inherit G-actions from X. Let $\gamma_Y^{\perp}$ denote the orthogonal complement of γ_Y in the product bundle $\varepsilon_Y(X)$ over Y with fiber X. Let $Z = (V, W)$ be a pair of real G-modules. We use the notation:

$$\begin{aligned}
\tau_Y &= \gamma_Y \otimes X, \\
\nu_Y^Z &= (\gamma_Y \otimes V_{\mathcal{L}(G)}) \oplus (\gamma_Y^{\perp} \otimes W), \\
\xi_Y^Z &= \tau_Y \oplus \nu_Y^Z.
\end{aligned}$$

By [10, Lemma 3.1], we have

$$\tau_Y \cong T(Y) \oplus \varepsilon_Y(\mathbb{R}) \tag{3.1}$$

as real G-vector bundles over Y, where $T(Y)$ is the tangent bundle of Y and $\varepsilon_Y(\mathbb{R})$ is the product line bundle.

Now suppose $G/G^{\mathrm{nil}} \cong C_3$ or C_6. Let $Z = (V, W)$ be a special 2-matched pair of real G-modules. Namely, we suppose $\dim V^G = \dim V^{G^{\{2\}}} = 1$, $\dim V^{G^{\{3\}}} = 3$, and $W^{G^{\{p\}}} = 0$ for all primes p. In particular, we have $V^{G^{\{3\}}} = V^{\mathcal{L}(G)}$ and $V_{G^{\{3\}}} = V_{\mathcal{L}(G)}$. Let M denote the real projective plane $P(V^{\mathcal{L}(G)})$. Then we have

$$\left[\mathrm{res}_P^G \xi_M^Z\right] = 0 \ \text{ in } \ \widetilde{KO}_P(\mathrm{res}_P^G M) \tag{3.2}$$

for all subgroups P whose order is a power of 2. Furthermore it follows from the next lemma that

$$\left[\mathrm{res}^G_Q \xi^Z_M\right] = 0 \ \text{ in } \ \widetilde{KO}_Q(\mathrm{res}^G_Q M)_{(q)} \tag{3.3}$$

for each subgroup Q whose order is a power of an odd prime q.

Lemma 3.2. *Let G be a cyclic group of odd order n and U a 3-dimensional real G-module. Let N be the real projective plane $P(U)$ associated with U and γ_N the canonical line bundle over N. Then the 4-fold Whitney sum ${\gamma_N}^{\oplus 4}$ of γ_N is G-isomorphic to the product bundle $\varepsilon_N(\mathbb{R}^4)$.*

If G is an Oliver group, then by using results in [11], [12], we obtain the following lemma from (3.2) and (3.3).

Lemma 3.3 (cf. [10, Lemma 4.5])**.** *Let G be an Oliver group such that $G/G^{\mathrm{nil}} \cong C_3$ or C_6 and $Z = (V, W)$ a special 2-matched pair of real G-modules. Let M be the real projective plane $P(V^{\mathcal{L}(G)})$. Then there exists a smooth G-action on a disk D possessing the following properties.*

(1) *$M \subset D$ as a G-submanifold and $D^G = M^G = \{x_0\}$.*

(2) *For each prime p, the set $D^L \smallsetminus M^L$, where $L = G^{\{p\}}$, is a closed subset of D.*

(3) *$T(D)|_M = T(M) \oplus \nu^Z_M \oplus \varepsilon_M(E)$ as real G-vector bundles, where E is an $\mathcal{L}(G)$-free real G-module.*

In Lemma 3.3, we note that

$$T_{x_0}(D) = {V^{\mathcal{L}(G)}}_G \oplus F \oplus E$$

where

$$F = V_{\mathcal{L}(G)} \oplus ({V^{\mathcal{L}(G)}}_G \otimes W). \tag{3.4}$$

This G-module F is $\mathcal{L}(G)$-free. If U is $\mathcal{P}$-matched to ${{V^{\mathcal{L}(G)}}_G}^{\oplus m} \oplus V_1$, then $U \oplus (F \oplus E)^{\oplus m}$ is $\mathcal{P}$-matched to $T_{x_0}(D)^{\oplus m} \oplus V_1$. Thus we obtain the following lemma.

Lemma 3.4 (cf. [10, Lemma 4.5])**.** *Let G, $Z = (V, W)$, M, D, E be as in the previous lemma. Further let U, V_1 and Ξ be $\mathcal{L}(G)$-free real G-modules, m a natural number, and N the m-fold cartesian product $M^{\times m}$ of M. Suppose U is $\mathcal{P}$-matched to ${{V^{\mathcal{L}(G)}}_G}^{\otimes m} \oplus V_1$. Then there exists a nonnegative integer K such that for any integer $k \geq K$, there exists a smooth G-action on a disk Δ possessing the following properties.*

(1) $N \amalg \{x_2\} \subset \Delta$ *as a G-submanifold and* $\Delta^G = \{x_1, x_2\}$*, where* $N^G = \{x_1\}$.

(2) *For each prime p, the set* $\Delta^L \smallsetminus (N^L \amalg \{x_2\})$*, where* $L = G^{\{p\}}$*, is a closed subset of* Δ.

(3) $T(\Delta)|_N = (T(M) \oplus \nu_M^Z \oplus \varepsilon_M(E))^{\times m} \oplus \varepsilon_N(V_1 \oplus \Xi \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus k})$ *as real G-vector bundles.*

(4) $T_{x_2}(\Delta) = U \oplus (F \oplus E)^{\oplus m} \oplus \Xi \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus k}$ *as real G-modules, where F is the G-module in* (3.4).

If G is a gap group in addition, we obtain the following lemma by using a G-surgery theorem in [9].

Lemma 3.5 (cf. [10, Lemma 4.5])**.** *Let* G*,* $Z = (V, W)$*,* M*,* D *and* E *be as in Lemma 3.3. Let* Ω *be a gap real G-module, m a natural number, N the m-fold cartesian product* $M^{\times m}$ *of M, and* Ξ *an* $\mathcal{L}(G)$*-free real G-module. Then there exists a nonnegative integer K such that for any integers a,* $b \geq K$*, there exists a smooth G-action on a sphere S possessing the following properties.*

(1) $N \subset S$ *as a G-submanifold and* $S^G = N^G = \{x_3\}$.

(2) *For each prime p, the set* $S^L \smallsetminus N^L$*, where* $L = G^{\{p\}}$*, is a closed subset of S.*

(3) $T(S)|_N = (T(M) \oplus \nu_M^Z \oplus \varepsilon_M(E))^{\times m} \oplus \varepsilon_N(\Xi \oplus \Omega^{\oplus a} \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus b})$ *as real G-vector bundles.*

Moreover, we can obtain the next lemma by using the same G-surgery theorem in [9].

Lemma 3.6. *Let G be a gap Oliver group,* Ω *a gap real G-module, and U an* $\mathcal{L}(G)$*-free real G-module. Then there exists a nonnegative integer K such that for any integers a,* $b \geq K$*, there exists a smooth G-action on a sphere* Σ *possessing the following properties.*

(1) $\Sigma^G = \{x_4\}$.

(2) *For each prime p, the set* $\Sigma^L \smallsetminus \{x_4\}$*, where* $L = G^{\{p\}}$*, is a closed subset of* Σ.

(3) $T_{x_4}(\Sigma) = U \oplus \Omega^{\oplus a} \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus b}$ *as real G-modules.*

As in [10], one can obtain Theorems 5 and 6 from Lemmas 3.4, 3.5 and 3.6.

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A survey of Borsuk-Ulam type theorems for isovariant maps

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ABSTRACT. In this article, we shall survey isovariant Borsuk-Ulam type theorems, which are interpreted as nonexistence results on isovariant maps from the viewpoint of equivariant topology or transformation group theory. We also discuss the existence problem of isovariant maps between representations as the converse of the isovariant Borsuk-Ulam theorem.

1 Introduction – backgrounds

Ever since K. Borsuk [5] proved the celebrated antipodal theorem, called the Borsuk-Ulam theorem, this theorem has attracted many researchers and has been generalized as Borsuk-Ulam type theorems, because it is not only beautiful but also has many interesting applications in several fields of mathematics like topology, nonlinear analysis and combinatorics. The original Borsuk-Ulam theorem is stated as follows:

Proposition 1.1. *For any continuous map $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$.*

Let C_2 be a cyclic group of order 2. Consider C_2-spheres S^m and S^n on which C_2 acts antipodally. By means of equivariant topology, the Borsuk-Ulam theorem is restated as follows:

Proposition 1.2. *If there is a C_2-map $f : S^m \to S^n$, then $m \leq n$. In other words, if $m > n$, then there is no C_2-map from S^m to S^n.*

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Thus, in the context of equivariant topology, Borsuk-Ulam type theorems are thought of as nonexistence results on G-maps. Such results are implicitly and explicitly applied in several mathematical problems; for example, a Borsuk-Ulam type result plays an important role in the proof of Furuta's 10/8-theorem [14] in 4-dimensional topology. Lovász [21] succeeded in proving Kneser's conjecture in combinatorics using the Borsuk-Ulam theorem. Matoušek [27] also illustrates several applications to combinatorics. Clapp [9] applied a Borsuk-Ulam type theorem to a certain problem in nonlinear analysis. Further results and applications on the Borsuk-Ulam theorem can be found in excellent survey articles [46], [47].

Wasserman [49] first considered an isovariant version of the Borsuk-Ulam theorem. Nagasaki [31], [33], [34] and Nagasaki-Ushitaki [38] also studied isovariant Borsuk-Ulam type theorems. For example, Wasserman's results imply the following.

Proposition 1.3. *Let G be a solvable compact Lie group and V, W real G-representations. If there exists a G-isovariant map from V to W, then the inequality*

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds. Here V^G denotes the G-fixed point set of V.

In this article, we shall survey isovariant Borsuk-Ulam type theorems and related topics, in particular, we shall discuss the existence or nonexistence problem for isovariant maps between G-representations or more general G-spaces. This article is organized as follows. In Section 2, after recalling some *equivariant* Borsuk-Ulam type theorems, we shall show the isovariant Borsuk-Ulam theorem for semilinear actions which is a generalization of Proposition 1.3. In Section 3, we shall discuss the question of determining for which groups the isovariant Borsuk-Ulam theorem holds. In Section 4, we shall discuss the existence of an isovariant map between representations. Finally, in Section 5, other topics, in particular, the isovariant Borsuk-Ulam theorem for pseudofree S^1-actions and its converse are discussed.

2 Isovariant Borsuk-Ulam type theorems

We first recall some definitions and notations in transformation group theory. Let G be a compact Lie group and X a G-space. For any $x \in X$, the isotropy subgroup G_x

at x is defined by $G_x = \{g \in G \mid gx = x\}$. We denote by $\mathrm{Iso}\, X$ the set of the isotropy subgroups. A subgroup of G always means a closed subgroup. The notation $H \leq G$ means that H is a subgroup of G, and $H < G$ means that H is a proper subgroup of G. As usual X^H denotes the H-fixed point set: $X^H = \{x \in X \mid hx = x\ (\forall h \in H)\}$. All G-equivariant maps (G-maps for short) are assumed to be continuous. A G-map $f : X \to Y$ is called *G-isovariant* if f preserves the isotropy subgroups, i.e., $G_x = G_{f(x)}$ for all $x \in X$. The notion of isovariance was introduced by Palais [41] in order to study a classification problem on orbit maps of G-spaces. Moreover isovariant maps often play important roles in classification problems of G-manifolds or equivariant surgery theory, for example, see [7], [44], [50]. For the study of these maps from the viewpoint of homotopy theory, see [12].

As mentioned in the Introduction, the Borsuk-Ulam theorem can be stated in the context of equivariant topology or transformation group theory. In this context, there are very rich researches and results on Borsuk-Ulam type theorems, see, for example, [1], [3], [4], [9], [11], [13], [15], [16], [17], [19], [20], [26], [43]. Here we present the following generalization of the Borsuk-Ulam theorem for free G-spaces.

Theorem 2.1 ([3], [37]). *Let C_k be a cyclic group of order k. Let X be an arcwise connected free C_k-space and Y a Hausdorff free C_k-space. If there exists a positive integer n such that $H_q(X; \mathbb{Z}/k) = 0$ for $1 \leq q \leq n$ and $H_{n+1}(Y/C_k; \mathbb{Z}/k) = 0$, then there is no continuous C_k-map from X to Y. Here this homology means the singular homology.*

Remark. This result can be deduced from a more general result of [3] and therein the Borel cohomology and spectral sequence arguments are used. On the other hand, in [37], the Smith homology is used. The advantage of the latter method is that the proof is still valid in the category of definable sets with the o-minimal structure over a real closed field, see [36], [37].

Theorem 2.1 implies a well-known Borsuk-Ulam type theorem.

Corollary 2.2 (mod p Borsuk-Ulam theorem). *Let C_p be a cyclic group of prime order p. Assume that C_p acts freely on mod p homology spheres Σ^m and Σ^n. If there is a C_p-map $f : \Sigma^m \to \Sigma^n$, then $m \leq n$. In other words, if $m > n$, then there is no C_p-map from Σ^m to Σ^n.*

In addition, one can see the following.

Corollary 2.3. *Let S^1 be a circle group. Assume that S^1 acts smoothly and fixed-point-freely on rational homology spheres Σ^m and Σ^n. If there is an S^1-map $f : \Sigma^m \to \Sigma^n$, then $m \leq n$.*

Proof. One can take a large prime number p such that C_p ($\leq S^1$) acts freely on Σ^m and Σ^n, and that Σ^m and Σ^n are mod p homology spheres. □

Thus Borsuk-Ulam type theorems are thought of as nonexistence results of G-maps. In this direction, we discuss the nonexistence of G-isovariant maps; namely, the isovariant Borsuk-Ulam theorem. We here recall a linear action or a homologically linear action on a (homology) sphere. Let V be a real representation of G, i.e., V is a (finite dimensional) real vector space on which G acts linearly. Since every representation of compact Lie group G is isomorphic to an orthogonal representation [2], we may suppose that the representation is orthogonal. Since, then, the G-action on V preserves the standard metric, it induces linear G-actions on the unit sphere SV and the unit disk DV. A G-manifold which is G-diffeomorphic to SV [resp. DV] is called a linear G-sphere [resp. a linear G-disk].

A *homologically linear* action on a homology sphere is defined as follows. Let G be a compact Lie group. Set

$$R_G = \begin{cases} \mathbb{Z}/|G| & \text{if } \dim G = 0 \\ \mathbb{Z} & \text{if } \dim G > 0. \end{cases}$$

Let Σ be an R_G-homology sphere, i.e., $H_*(\Sigma; R_G) \cong H_*(S^m; R_G)$, where $m = \dim \Sigma$. Suppose that G acts smoothly on Σ.

Definition.

(1) The G-action on Σ is called *homologically linear* if for every subgroup H of G, the H-fixed point set Σ^H is an R_G-homology sphere or the empty set.

(2) The G-action on Σ is called *semilinear* or *homotopically linear* if for every subgroup H of G, the H-fixed point set Σ^H is a homotopy sphere or the empty set. (Hence Σ itself must be a homotopy sphere.)

(3) We call a smooth closed manifold Σ with homotopically linear [resp. semilinear] G-action a homotopically linear [resp. semilinear] G-sphere.

Let $\mathcal{H}_G$ denote the family of homologically linear G-spheres and $\mathcal{S}_G$ the family of semilinear G-spheres. We also denote by $\mathcal{L}_G$ the family of linear G-spheres.

Remark. Clearly $\mathcal{L}_G \subset \mathcal{S}_G \subset \mathcal{H}_G$, but the converse inclusions are not true in general. For semilinear actions on spheres, see [29], [30], [32].

Lemma 2.4. *Let $\mathcal{F}_G = \mathcal{H}_G$, $\mathcal{S}_G$ or $\mathcal{L}_G$.*

(1) *Let H be a subgroup of G. If $\Sigma \in \mathcal{F}_G$, then $\Sigma \in \mathcal{F}_H$ by restriction of the action.*

(2) *Let H be a normal subgroup of G. If $\Sigma \in \mathcal{F}_G$, then $\Sigma^H \in \mathcal{F}_{G/H}$.*

Proof. (1) Since Σ^K, $K \leq H$, is an R_G-homology sphere (or the empty set), it is also an R_H-homology sphere (or the empty set).

(2) Since $(\Sigma^H)^{K/H} = \Sigma^K$ is an R_G-homology sphere (or the empty set), it is also an $R_{G/H}$-homology sphere (or the empty set). □

Now we state the isovariant Borsuk-Ulam theorem for homologically linear actions.

Theorem 2.5 (Isovariant Borsuk-Ulam theorem)**.** *Let G be a solvable compact Lie group. If there is a G-isovariant map $f : \Sigma_1 \to \Sigma_2$ between homologically linear G-spheres Σ_i, $i = 1, 2$, then the inequality*

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G$$

holds.

A convention: if Σ_i^G is empty, then we set $\dim \Sigma_i^G = -1$. To prove the theorem, we make the following definition.

Definition. We say that G has the IB-property in $\mathcal{F}_G$, where $\mathcal{F}_G = \mathcal{L}_G$, $\mathcal{S}_G$ or $\mathcal{H}_G$ if G has the following property: If there is a G-isovariant map $f : \Sigma_1 \to \Sigma_2$, $\Sigma_i \in \mathcal{F}_G$, then the inequality

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G$$

holds.

We first show the following fact.

Lemma 2.6 ([49], [31])**.**

(1) *Let H be a normal subgroup of G. If H and G/H have the IB-properties in $\mathcal{F}_H$ and $\mathcal{F}_{G/H}$ respectively, then G also has the IB-property in $\mathcal{F}_G$.*

(2) *Let $\mathcal{F}_G = \mathcal{S}_G$ or $\mathcal{L}_G$. If G has the IB-property in $\mathcal{F}_G$, then G/H also has the IB-property in $\mathcal{F}_{G/H}$.*

Proof. (1) Let $f : \Sigma_1 \to \Sigma_2$ be any G-isovariant map between Σ_1 and Σ_2 in $\mathcal{F}_G$. Then $\mathrm{res}_H f : \Sigma_1 \to \Sigma_2$ is H-isovariant and $f^H : \Sigma_1^H \to \Sigma_2^H$ is G/H-isovariant. It follows from Lemma 2.4 that Σ_1, $\Sigma_2 \in \mathcal{F}_H$ and Σ_1^H, $\Sigma_2^H \in \mathcal{F}_{G/H}$. By assumption, we have

$$\dim \Sigma_1 - \dim \Sigma_1^H \leq \dim \Sigma_2 - \dim \Sigma_2^H,$$

$$\dim \Sigma_1^H - \dim \Sigma_1^G \leq \dim \Sigma_2^H - \dim \Sigma_2^G.$$

Hence we obtain

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G.$$

(2) Suppose that $f : \Sigma_1 \to \Sigma_2$ is a G/H-isovariant map between Σ_1 and $\Sigma_2 \in \mathcal{F}_{G/H}$. Via the projection $G \to G/H$, the G/H-action lifts to a G-action. Hence Σ_i, $i = 1, 2$, are thought of as in $\mathcal{F}_G$ and then f is a G-isovariant map. Thus we have $\dim \Sigma_1 - \dim \Sigma_1^{G/H} \leq \dim \Sigma_2 - \dim \Sigma_2^{G/H}$, since $\dim \Sigma_i^{G/H} = \dim \Sigma_i^G$. □

Proof of Theorem 2.5. We show that a solvable compact Lie group has the IB-property in $\mathcal{H}_G$. Suppose that $f : \Sigma_1 \to \Sigma_2$ is a G-isovariant map. Since G is solvable, there is a normal series of closed subgroups:

$$1 = H_0 \lhd H_1 \lhd \cdots \lhd H_r = G$$

such that H_i/H_{i-1}, $1 \leq i \leq r$, is isomorphic to C_p (p: some prime) or S^1.

By Lemmas 2.4 and 2.6, the proof is reduced to the cases of C_p and S^1; moreover, the case of S^1 is also reduced to the case of C_p, since there exists some cyclic subgroup C_p of S^1 such that $\Sigma_i^{S^1} = \Sigma_i^{C_p}$, $i = 1, 2$, in fact, $\Sigma_i \in \mathcal{H}_G$ has only finitely many orbit types [6], [18].

In the case of C_p, the proof proceeds as follows. Let $G = C_p$. Since f is G-isovariant, it follows that $f(\Sigma_1 - \Sigma_1^G) \subset \Sigma_2 - \Sigma_2^G$. Set $N_i := \Sigma_i - \Sigma_i^G$. Since Σ_i and Σ_i^G

are mod p homology spheres, by Alexander duality, one can see that $N_i := \Sigma_i - \Sigma_i^G$ has the same mod p homology groups as a sphere S^{n_i}, where $n_i = \dim \Sigma_i - \dim \Sigma_i^G - 1$. Since $G = C_p$ acts freely on N_i and $f|_{N_1} : N_1 \to N_2$ is a G-map, it follows from Theorem 2.1 that $n_1 \leq n_2$, and hence

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G.$$

Thus C_p has the IB-property in $\mathcal{H}_{C_p}$. □

This theorem implies Proposition 1.3.

Proof of Proposition 1.3. Let $f : V \to W$ be a G-isovariant map between representations. Let $V^{G\perp}$ denote the orthogonal complement of V^G in V, and then V decomposes as $V = V^G \oplus V^{G\perp}$. Similarly W decomposes as $W = W^G \oplus W^{G\perp}$. Then the composition map $g := p \circ f \circ i : V^{G\perp} \to W^{G\perp}$ is a G-isovariant map, where $i : V^{G\perp} \to V$ is the inclusion and $p : W \to W^{G\perp}$ is the projection. Since $g^{-1}(0) = \{0\}$, g induces a G-isovariant map $g_0 : V^{G\perp} \setminus \{0\} \to W^{G\perp} \setminus \{0\}$. By normalizing, one has a G-isovariant map $g_1 : S(V^{G\perp}) \to S(W^{G\perp})$. Since G has the IB-property in $\mathcal{H}_G$, one has

$$\dim S(V^{G\perp}) + 1 \leq \dim S(W^{G\perp}) + 1,$$

which leads to the inequality

$$\dim V - \dim V^G \leq \dim W - \dim W^G.$$

□

The following is obtained from Smith theory [6], [18].

Corollary 2.7. *Let G be a finite p-group. Let Σ_i, $i = 1, 2$, be mod p homology spheres with G-actions. If there is a G-isovariant map $f : \Sigma_1 \to \Sigma_2$, then*

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G$$

holds.

Proof. Smith theory shows that for every subgroup H, the H-fixed point set Σ_i^H is a mod p homology sphere or the empty set. Hence the G-action on Σ_i is homologically linear and $\Sigma_i \in \mathcal{H}_G$. □

The singular set $N^{>1}$ of a G-manifold N is defined by

$$N^{>1} = \bigcup_{1 \neq H \leq G} N^H.$$

The following is a variant of the isovariant Borsuk-Ulam theorem, which is a generalization of a result in [33].

Corollary 2.8. *Let G be a finite group and W a representation of G. Let M and N be homologically linear G-spheres and assume that G acts freely on M. If there is a G-isovariant map $f : M \to N$, then the inequality*

$$\dim M + 1 \leq \dim N - \dim N^{>1}$$

holds.

Proof. Since $\dim N^{>1} = \max\{\dim N^H \mid 1 \neq H \leq G\}$, there is a subgroup $H \neq 1$ such that $\dim N^H = \dim N^{>1}$. Taking a cyclic subgroup $C_p \leq H$ of prime order, one has $\dim N^{C_p} = \dim N^{>1}$. By restricting to the C_p-action, it turns out that f is a C_p-isovariant map. Hence, by the isovariant Borsuk-Ulam theorem, one has

$$\dim M + 1 \leq \dim N - \dim N^{C_p} = \dim N - \dim N^{>1}.$$

□

Remark. Not all finite groups can act freely on a (homology) sphere. For details, see [8], [23], [24], [25].

3 Which groups have the IB-property?

As seen in the previous section, a solvable compact Lie group has the IB-property in $\mathcal{F}_G$, i.e., the isovariant Borsuk-Ulam theorem holds in $\mathcal{F}_G$. In this section, we discuss the question: Which compact Lie groups have the IB-property in $\mathcal{F}_G$? First we consider the case of $\mathcal{F}_G = \mathcal{H}_G$ or $\mathcal{S}_G$. In this case, a complete answer is known.

Theorem 3.1 (cf. [31])**.** *The following statements are equivalent.*

(1) *G has the IB-property in $\mathcal{H}_G$.*

(2) *G has the IB-property in $\mathcal{S}_G$.*

(3) *G is a solvable compact Lie group.*

Proof. We have already seen the implication (3) $\Rightarrow$ (1), and trivially (1) $\Rightarrow$ (2). To see (2) $\Rightarrow$ (3), we show that there is a counterexample to the isovariant Borsuk-Ulam theorem when G is nonsolvable. According to Oliver [40, Theorem 4], there exists a disk D with a smooth G-action such that D^H is also a disk if H is a solvable subgroup, and the empty set if H is a nonsolvable subgroup. The boundary of this G-disk D is clearly a semilinear G-sphere. Note also that $D^G = \emptyset$ since G is nonsolvable. Set $\Sigma_n = \partial(D \times D(\mathbb{R}^n))$, where $D(\mathbb{R}^n)$ is an n-dimensional disk with trivial G-action. Then Σ_n is a semilinear G-sphere without G-fixed points. For any positive integer n, take a map $h_n : D(\mathbb{R}^n) \to D(\mathbb{R}^1)$ such that $h_n(D(\mathbb{R}^n)) \subset \partial D(\mathbb{R}^1)$, and define a G-map

$$g_n := id \times h_n : D \times D(\mathbb{R}^n) \to D \times D(\mathbb{R}^1).$$

Then one can easily see that g_n is a G-isovariant map and g_n maps the boundary $\partial(D \times D(\mathbb{R}^n))$ into the boundary $\partial(D \times D(\mathbb{R}^1))$. Hence we obtain a G-isovariant map $f_n := g_{n|\Sigma_n} : \Sigma_n \to \Sigma_1$. Since $\dim \Sigma_n > \dim \Sigma_1$ for $n > 1$, this f_n gives a counterexample to the isovariant Borsuk-Ulam theorem. $\square$

Remark. The semilinear G-sphere Σ_1 in the above proof is equivariantly embedded in some linear G-sphere SW [6]. Hence there is an isovariant map $f_n : \Sigma_n \to SW$ such that $\dim \Sigma_n + 1 > \dim SW - \dim SW^G$ for some large n.

In the case $\mathcal{F}_G = \mathcal{L}_G$, the problem is more difficult; in fact, a complete answer is unfortunately unknown to the best of the author's knowledge. However some partial answers are known. We present them here without proof.

To state Wasserman's result, we recall the *prime condition* for a finite group G.

Definition. We say that a finite simple group G satisfies the prime condition if for every element $g \in G$,

$$\sum_{p|o(g)} \frac{1}{p} \leq 1$$

holds, where $o(g)$ denotes the order of g, and p is a prime dividing $o(g)$.

We say that a finite group G satisfies the prime condition if every simple factor group in a normal series of G satisfies the prime condition as a simple group.

Theorem 3.2 ([49])**.** *Every finite group satisfying the prime condition has the IB-property in* $\mathcal{L}_G$*.*

This theorem provides nonsolvable examples of having the IB-property in $\mathcal{L}_G$.

Example 3.3. The alternating groups $A_5, A_6, \ldots, A_{11}$ satisfy the prime condition, and hence A_i has the IB-property in $\mathcal{L}_{A_i}$, $i = 5, 6, \ldots, 11$.

Remark. The alternating groups A_n, $n > 11$, do not satisfy the prime condition. However, the author does not know whether A_n has the IB-property for $n > 11$.

Another partial answer is a weak version of the isovariant Borsuk-Ulam theorem.

Theorem 3.4 ([31])**.** *For an arbitrary compact Lie group* G*, there exists a weakly monotone increasing function* $\varphi_G : \mathbb{N}_0 \to \mathbb{N}_0$ *diverging to* ∞ *with the following property.*

(WIB) *For any pair of representations* V *and* W *such that there is a* G*-isovariant map* $f : SV \to SW$*, the inequality*

$$\varphi_G(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

holds.

Here $\mathbb{N}_0$ *denotes the set of nonnegative integers.*

The above result does not hold for G-equivariant maps even if $SV^G = SW^G = \emptyset$. For example, when G is a cyclic group C_{pq} of order pq, where p, q are distinct primes, a Borsuk-Ulam type theorem does not hold as can be seen below.

Let U_k $(= \mathbb{C})$ be the representation of $C_n = \langle g \rangle$ for which g acts by $g \cdot z = \xi^k z$, $z \in U_k$, where $\xi = \exp(2\pi\sqrt{-1}/n)$.

Proposition 3.5 (cf. [48])**.** *Let* $C_{pq} = \langle g \rangle$ *be a cyclic group of order* pq*, where* p*,* q *are distinct primes. For any positive integer* r*, there is a* C_{pq}*-map* $f : S(rU_1) \to S(U_p \oplus U_q)$*, where* rU_1 *is the direct sum of* r *copies of* U_1*.*

Proof. Set $G = C_{pq}$. By a result of [48], there is a self G-map $h : S(U_p \oplus U_q) \to S(U_p \oplus U_q)$ with $\deg h = 0$; hence h is (nonequivariantly) nullhomotopic. Since G acts freely on $S(rU_1)$, $S(rU_1)$ has a G-CW complex structure consisting of free G-cells. We put $S(rU_1) = \bigcup_k X_k$, where X_k is the k-skeleton. A G-map from $S(rU_1)$

to $S(U_p \oplus U_q)$ is inductively constructed as follows. Suppose that one has a G-map $f_{k-1} : X_{k-1} \to S(U_p \oplus U_q)$. Then f_{k-1} can be extended to a G-map $f' : X_{k-1} \cup_\phi G \times D^k \to S(U_p \oplus U_q)$. Indeed, since $h \circ \phi_{|\{1\} \times S^{k-1}} : S^{k-1} \to S(U_p \oplus U_q)$ is nullhomotopic, $h \circ \phi_{|S^{k-1}}$ is extended to a map $g : D^k \to S(U_p \oplus U_q)$, and next g is equivariantly extended to a G-map $g' : G \times D^k \to S(U_p \oplus U_q)$. By gluing g' to f_{k-1}, one obtains a G-map $f' : X_{k-1} \cup_\phi G \times D^k \to S(U_p \oplus U_q)$. Repeating this procedure, one has a G-map $f_k : X_k \to S(U_p \oplus U_q)$. □

Remark. More generally, Bartsch [1] shows that a weak version of the Borsuk-Ulam theorem holds for linear G-spheres of a finite group G if and only if G has prime power order.

Combining this proposition with the isovariant Borsuk-Ulam theorem, we obtain another Borsuk-Ulam type result.

Corollary 3.6. *For any C_{pq}-map $f : S(rU_1) \to S(U_p \oplus U_q)$, $r \geq 2$, the image of f meets the Hopf link $SU_p \coprod SU_q$ in $S(U_p \oplus U_q)$.*

Proof. Suppose that $f^{-1}(SU_p \coprod SU_q) = \emptyset$. Then f is a G-isovariant map, since G acts freely on $S(U_p \oplus U_q) \setminus (SU_p \coprod SU_q)$. Furthermore $\mathrm{Res}_{C_p} f$ is a C_p-isovariant map. By the isovariant Borsuk-Ulam theorem, it follows that

$$2r = \dim S(rU_q) + 1 \leq \dim S(U_p \oplus U_q) - \dim S(U_p \oplus U_q)^{C_p} = 2.$$

This is a contradiction. □

Remark. Another equivalent statement of the original Borsuk-Ulam theorem is: For any C_2-map $f : S^n \to \mathbb{R}^n$, the image of f meets the origin in $\mathbb{R}^n$, where the C_2-actions on S^n and $\mathbb{R}^n$ are both given by multiplication by -1.

4 The converse of the isovariant Borsuk-Ulam theorem

The isovariant Borsuk-Ulam theorem is interpreted as a nonexistence result of isovariant maps, and it produces several inequalities, which give a necessary condition for the existence of an isovariant map. In several cases, it is also sufficient. In this section, we shall present such examples for the existence of an isovariant map and discuss the converse of the isovariant Borsuk-Ulam theorem. The materials are taken from [34], [35], [38], [39].

As a special case of Corollary 2.8, we consider the case $N = SW$, a linear G-sphere. Then, the inequality

$$\dim M + 1 \leq \dim SW - \dim SW^{>1}$$

holds if there is a G-isovariant map $f : M \to SW$. In this case, we show that the converse is true.

Proposition 4.1. *Let G be a finite group and M a compact G-manifold with free G-action. Let W be a representation of G. If $\dim M + 1 \leq \dim SW - \dim SW^{>1}$, then there exists a G-isovariant map $f : M \to SW$.*

We define the free part SW_{free} of SW by $SW_{\text{free}} = SW \setminus SW^{>1}$. Set $d = \dim SW - \dim SW^{>1}$.

Lemma 4.2. *The free part SW_{free} is $(d-2)$-connected.*

Proof. Since $\dim S^k \times I + \dim SW^{>1} < \dim SW$ for $k \leq d-2$, any homotopy into SW deforms to a homotopy into SW_{free} by a general position argument. Hence every map from S^k to SW_{free} is nullhomotopic for $k \leq d-2$. □

Proof of Proposition 4.1. Since G acts freely on M and SW_{free}, it suffices to show the existence of a G-map from M to SW_{free}. Since M has a G-CW complex structure, we may put $M = \bigcup_k X_k$, where X_k is the k-skeleton of M. The inequality means $k \leq d-1$. A G-map into SW_{free} is inductively constructed as follows. Suppose that f_{k-1} is constructed as a G-map from X_{k-1} to SW_{free}; then f_{k-1} is extended on $X_{(k-1)} \cup_\phi G \times D^k$; indeed, since SW_{free} is $(d-2)$-connected, the map $f_{k-1} \circ \phi_{|\{1\} \times S^{k-1}} : S^{k-1} \to SW_{\text{free}}$ is extended to a map $g : D^k \to SW_{\text{free}}$ and then g is equivariantly extended to a G-map $\tilde{g} : G \times D^k \to SW_{\text{free}}$. By gluing $\tilde{g}$ to f_{k-1}, one can obtain a G-map $f'_{k-1} : X_{k-1} \cup_\phi G \times D^k \to SW_{\text{free}}$. Repeating this procedure, we obtain a G-map from the k-skeleton X_k to SW_{free}, and finally we obtain a G-map $f : M \to SW_{\text{free}}$. □

Thus, in this situation, the existence problem is solved as follows.

Corollary 4.3. *Let M be a mod $|G|$ homology sphere with free G-action and W a representation of G. There exists a G-isovariant map from M to SW if and only if*

$$\dim M + 1 \leq \dim SW - \dim SW^{>1}.$$

Next we consider the existence problem of an isovariant map between (real) representations. Let G be a finite solvable group. Let $f : V \to W$ be a G-isovariant map between representations V and W. Take any pair (H, K) of subgroups of G with $H \lhd K$. Then $f^H : V^H \to W^H$ is considered as a K/H-isovariant map. Since K/H is solvable, the isovariant Borsuk-Ulam theorem implies the inequality

$$\dim V^H - \dim V^K \leq \dim W^H - \dim W^K.$$

From this observation, we consider the following condition for a pair of representations V and W of a solvable group G:

$(C_{V,W})$ $\dim V^H - \dim V^K \leq \dim W^H - \dim W^K$ for every pair (H, K) with $H \lhd K$.

Moreover the condition $(I_{V,W})$: $\mathrm{Iso}\, V \subset \mathrm{Iso}\, W$ is obviously necessary for the existence of an isovariant map.

Definition. We say that a finite solvable group G has the *complete IB-property* if, for every pair (V, W) of representations satisfying conditions $(C_{V,W})$ and $(I_{V,W})$, there exists a G-isovariant map from V to W.

Remark. If G is nilpotent, then $(C_{V,W})$ implies $(I_{V,W})$ [34].

Which solvable groups have the complete IB-property? A complete answer is not known; however, some examples that have the complete IB-property are known.

Theorem 4.4 ([34], [35]**).** *Let p, q, r be distinct primes. The following finite groups have the complete IB-property:*

(1) *abelian p-groups,*

(2) *$C_{p^m q^n}$: cyclic groups of order $p^m q^n$,*

(3) *C_{pqr}: cyclic groups of order pqr,*

(4) *D_3, D_4 and D_6: dihedral groups of order 6, 8 and 12, respectively.*

Remark. S. Kôno also obtains a similar result in the case of complex $C_{p^m q^n}$-representations.

For details of the proof, see [34] and [35]. The idea is to decompose (V, W) into primitive pairs (V_i, W_i).

Definition. A pair of representations (V, W) is called *primitive* if V and W cannot be decomposed into $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$ such that (V_i, W_i), $V_i \neq 0$, $W_i \neq 0$, satisfies (C_{V_i,W_i}), $i = 1, 2$.

Then, by constructing a G-isovariant map $f_i : V_i \to W_i$, we have a G-isovariant map $f = \oplus_i f_i : V \to W$.

Example 4.5. The following are examples of primitive pairs of C_n-representations, and there exist isovariant maps between the representations. Suppose that p, q, r are pairwise coprime integers greater than 1.

(1) (U_k, U_l) when $(k, n) = (l, n) = 1$.

(2) $(U_1, U_p \oplus U_q)$ when pq divides n.

(3) $(U_p \oplus U_q,\ U_{p^2} \oplus U_{pq})$ when $p^2 q$ divides n.

(4) $(U_p \oplus U_q \oplus U_r,\ U_1 \oplus U_{pq} \oplus U_{qr} \oplus U_{pr})$ when pqr divides n.

In the cases (1)-(3), one can define a C_n-isovariant map concretely; however, in case (4), equivariant obstruction theory is used. We illustrate it in Section 5.

On the other hand, there exists a group not having the complete IB-property.

Theorem 4.6 ([35])**.** *Let D_n be the dihedral group of order $2n$ $(n \geq 3)$. Every D_n $(n \neq 3, 4, 6)$ does not have the complete IB-property.*

The dihedral group D_n has the following presentation:

$$D_n = \langle a, b \,|\, a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle.$$

One has the normal cyclic subgroup $C_m = \langle a^{n/m} \rangle$ of D_n for every divisor m of n, and there are n/m dihedral subgroups $\langle a^{n/m}, b \rangle, \langle a^{n/m}, b^2 \rangle, \ldots, \langle a^{n/m}, a^{n/m-1} b \rangle$ containing C_m as a subgroup of index 2. If n/m is odd, then these are all conjugate in D_n. As a representative of their conjugacy class, we take $D_m = \langle a^{n/m}, b\rangle$. If n/m is even, then there are two conjugacy classes. As representatives, we take $D_m = \langle a^{n/m}, b\rangle$ and $D'_m = \langle a^{n/m}, ab\rangle$.

Let $T_k = \mathbb{C}$, $1 \le k < n/2$, be the D_n-representation on which D_n acts by $a \cdot z = \xi^k z$, $b \cdot z = \overline{z}$, $z \in S_k$, where $\xi = \exp(2\pi\sqrt{-1}/n)$. These T_k are all (nonisomorphic) 2-dimensional irreducible representations over $\mathbb{R}$ [45]. It follows that Ker $T_k = C_{(k,n)}$. and

$$\text{Iso } T_k = \{D_n, \langle a^{n/(k,n)}, a^t b\rangle, \langle a^{n/(k,n)}\rangle \mid 0 \le t \le n-1\}.$$

Note also that

$$\dim T_k^H = \begin{cases} 2 \text{ if } H \le C_{(k,n)} \\ 1 \text{ if } H \text{ is conjugate to } D_{(k,n)} \text{ or } D'_{(k,n)} \\ 0 \text{ otherwise.} \end{cases}$$

Proof of Theorem 4.6. Let k be an integer prime to n with $1 < k < n/2$. Consider a pair $(T_1,\ T_k)$ of representations of D_n. It is easily seen that $(T_1,\ T_k)$ satisfies conditions (C_{T_1,T_k}) and (I_{T_1,T_k}). We show that there is no D_n-isovariant map from T_1 to T_k. Suppose that there is a D_n-isovariant map from T_1 to T_k for some k; then, by normalization, one has a D_n-isovariant map $f : ST_1 \to ST_k$. Note that $ST_1^{>1} = ST_k^{>1} = \{\exp(\pi t\sqrt{-1}/n) \mid 0 \le t \le n-1\}$. Take $x = 1$ and $y = \exp(\pi\sqrt{-1}/n)$, then the isotropy subgroup at x in ST_1 is $\langle b\rangle$, and also the isotropy subgroup at y in ST_1 is $\langle ab\rangle$. Since $ST_k^{\langle b\rangle} = \{\pm 1\} \subset \mathbb{C}$, it follows that $f(1) = \pm 1$. Composing, if necessary, the antipodal map $z \mapsto -z$ on ST_k with f, we may assume $f(1) = 1$. Let A be the shorter arc joining x with y in ST_1. Since every point of the interior of A has trivial isotropy subgroup, it follows that $f(A \setminus \{x, y\})$ is contained in $ST_k \setminus ST_k^{>1}$; hence $f(y)$ must be y or $\overline{y}$. However the isotropy subgroup at y (resp. $\overline{y}$) in ST_k is equal to $\langle a^r b\rangle$ (resp. $\langle a^{-r} b\rangle$), where r is a positive integer with $kr \equiv 1 \bmod n$, but it is not equal to $\langle ab\rangle$, since $k \not\equiv \pm 1 \bmod n$. This contradicts the isovariance of f. Thus the proof is complete. □

5 The existence of isovariant maps from a rational homology sphere with pseudofree S^1-action to a linear S^1-sphere

Let $G = S^1$ ($\subset \mathbb{C}$). Let T_i ($= \mathbb{C}$) be the irreducible representation of S^1 defined by $g \cdot z = g^k z$. Let M be a rational homology sphere with *pseudofree* S^1-action.

Definition (Montgomery-Yang [28])**.** An S^1-action on M is *pseudofree* if

(1) the action is effective, and

(2) the singular set $M^{>1} := \bigcup_{1 \neq H \leq S^1} M^H$ is not empty and consists of finitely many exceptional orbits.

Here an orbit $G(x)$ is called exceptional if $G(x) \cong S^1/D$, $(1 \neq D < S^1)$ [6].

Remark. Other meanings for the term "pseudofree action" appear in the literature.

Example 5.1. Let $V = T_p \oplus T_q \oplus T_r$. Then the S^1-action on SV is pseudofree. Indeed it is clearly effective, and

$$\begin{aligned} SV^{>1} =& ST_p \coprod ST_q \coprod ST_r \\ \cong& S^1/C_p \coprod S^1/C_q \coprod S^1/C_r. \end{aligned}$$

Remark. There are many "exotic" pseudofree S^1-actions on high-dimensional homotopy spheres [28], [42].

Then the following isovariant Borsuk-Ulam type result can be verified.

Theorem 5.2 ([33])**.** *Let M be a rational homology sphere with* pseudofree *S^1-action and SW a linear S^1-sphere. There is an S^1-isovariant map $f : M \to SW$ if and only if*

(I): $\operatorname{Iso} M \subset \operatorname{Iso} SW$,

(PF1): $\dim M - 1 \leq \dim SW - \dim SW^H$ *when H is a nontrivial subgroup which is contained in some $D \in \operatorname{Iso} M$,* (PF2): $\dim M + 1 \leq \dim SW - \dim SW^H$ *when* H *is a nontrivial subgroup which is not contained in any* $D \in \operatorname{Iso} M$.

We give some examples. Let p, q, r be pairwise coprime integers greater than 1.

Example 5.3. There is no S^1-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Proof. Condition (PF1) is not fulfilled. □

Remark. There is an S^1-equivariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Example 5.4. There is an S^1-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Proof. One can see that $\operatorname{Iso} M = \{1, C_p, C_q, C_r\}$ and

$$\operatorname{Iso} SW = \{1, C_p, C_q, C_r, C_{pq}, C_{qr}, C_{rp}\}.$$

Hence it is easily seen that (PF1) and (PF2) are fulfilled and $\operatorname{Iso} M \subset \operatorname{Iso} SW$. By the theorem above, there is an S^1-isovariant map. □

From this, we obtain an isovariant map in the case of Example 4.5(4).

Corollary 5.5. *There is an* C_{pqr}*-isovariant map*

$$f : S(U_p \oplus U_q \oplus U_r) \to S(U_1 \oplus U_{pq} \oplus U_{qr} \oplus U_{rp}).$$

Proof. By restricting f in Example 5.4 to the C_{pqr}-action, one has the desired map. □

5.1 Proof of Theorem 5.2 (outline)

We shall give an outline of the proof of Theorem 5.2. Full details can be found in [33]. Set $SW_{\text{free}} := SW \setminus SW^{>1}$. Note that S^1 acts freely on SW_{free}. Let N_i be an S^1-tubular neighborhood of each exceptional orbit in M. By the slice theorem, N_i is identified with $S^1 \times_{D_i} DU_i$ $(1 \leq i \leq r)$, where D_i is the isotropy group of the exceptional orbit and U_i is the slice D_i-representation. Set $X := M \setminus (\coprod_i \operatorname{int} N_i)$. Note that S^1 acts freely on X.

The "only if" part is proved by the (isovariant) Borsuk-Ulam theorem. Indeed for (PF1), take a point $x \in M$ with $G_x = D$ and a D-invariant closed neighborhood B of x which is D-diffeomorphic to some unit disk DV. Hence we obtain an H-isovariant map $f_{|SV} : SV \to SW$ by restriction. Applying the isovariant Borsuk-Ulam theorem to f, we obtain (PF1).

We next show (PF2). Since f is isovariant, one sees that f maps M into $SW \setminus SW^H$. Since $SW \setminus SW^H$ is S^1-homotopy equivalent to $S(W^{H^\perp})$, one obtains an S^1-map $g : M \to S(W^{H^\perp})$. By Corollary 2.3, condition (PF2) follows.

To show the converse, we use the equivariant obstruction theory. We recall the following result.

Lemma 5.6. *There is an S^1-isovariant map $\tilde{f}_i : N_i \to SW$.*

Proof. Let $N_i = N \cong_G S^1 \times_D DV \subset M$, where D is the isotropy group of the exceptional orbit and V is the slice representation. Similarly take a closed S^1-tubular neighborhood N' of an exceptional orbit with isotropy group D, and set $N' \cong_G S^1 \times_D DV' \subset SW$. By (PF1), one sees that $\dim SV + 1 \leq \dim SV' - \dim SV'^{>1}$. Since D acts freely on SV, there is a D-map $g : SV \to SV' \setminus SV^{>1} \subset SW$ by Corollary 2.8, which leads to a D-isovariant map $g : SV \to SW$. Taking a cone, we have a D-isovariant map $\tilde{g} : DV \to DV'$, and hence an S^1-isovariant map $\overline{f} = S^1 \times_D \tilde{g} : N \to N' \subset SW$. □

Set $f_i := \tilde{f}_i|_{\partial N_i} : \partial N_i \to SW_{\text{free}}$, and $f := \coprod_i f_i : \partial X \to SW_{\text{free}}$. If f is extended to an S^1-map $F : X \to SW_{\text{free}}$, by gluing the maps, we obtain an S^1-isovariant map

$$F \cup (\coprod_i \tilde{f}_i) : M \to SW.$$

Thus we need to investigate the extendability of an S^1-map $f : \partial X \to SW_{\text{free}}$ to $F : X \to SW_{\text{free}}$. Equivariant obstruction theory [10] answers this question. A standard computation shows

Lemma 5.7 ([33], [38]**).** *Set* $d = \dim SW - \dim SW^{>1}$.

(1) SW_{free} *is* $(d-2)$*-connected and* $(d-1)$*-simple.*

(2) $\pi_{d-1}(SW_{\text{free}}) \cong H_{d-1}(SW_{\text{free}})) \cong \oplus_{H \in \mathcal{A}} \mathbb{Z}$, *where*

$$\mathcal{A} := \{H \in \operatorname{Iso} SW \,|\, \dim SW^H = \dim SW^{>1}\}$$

and the generators are represented by $S(W^{H^\perp})$, $H \in \mathcal{A}$.

By noticing that $\dim M - 1 \leq d$ by (PF1) and (PF2), the obstruction $\mathfrak{o}_{S^1}(f)$ to the existence of an S^1-map $F : X \to SW_{\text{free}}$ lies in the equivariant cohomology group

$$\mathfrak{H}^d_{S^1}(X, \partial X; \pi_{d-1}(SW_{\text{free}})) \cong H^d(X/S^1, \partial X/S^1; \pi_{d-1}(SW_{\text{free}})).$$

If $\dim M - 1 < d$ (i.e., $\dim X/S^1 < d$), then one sees that

$$H^*(X/S^1, \partial X/S^1; \pi_{*-1}(SW_{\text{free}})) = 0$$

by dimensional reasons. Hence the obstruction vanishes and there exists an extension $F : X \to SW_{\text{free}}$.

We hereafter assume that $\dim M - 1 = d$ (i.e., $\dim X/S^1 = d$). The computation of the obstruction is executed by the multidegree.

Definition. Let $N = S^1 \times_D DU \subset M$, $1 \neq D \in \operatorname{Iso} M$. Assume that $\dim M - 1 = \dim U = d$. Let $f : \partial N \to SW_{\text{free}}$ be an S^1-map, and consider the D-map $\overline{f} = f|_{SU} : SU \to SW_{\text{free}}$. Then the multidegree of f is defined by

$$\operatorname{mDeg} f := \overline{f}_*([SU]) \in \oplus_{H \in \mathcal{A}} \mathbb{Z},$$

under the natural identification $H_{d-1}(SW_{\text{free}})$ with $\oplus_{H \in \mathcal{A}} \mathbb{Z}$.

The obstruction $\mathfrak{o}_{S^1}(f)$ is described by the multidegree as follows.

Proposition 5.8 ([33])**.** *Let $F_0 : X \to SW_{\text{free}}$ be a fixed S^1-map; this map always exists, however, it is not necessary to extend it to an isovariant map on M. Set $f_{0,i} = F_0|_{\partial N_i}$. Then*

$$\mathfrak{o}_{S^1}(f) = \sum_{i=1}^{r} (\text{mDeg}\, f_i - \text{mDeg}\, f_{0,i})/|D_i|,$$

under the natural identification $H_{d-1}(SW_{\text{free}})$ with $\oplus_{H \in \mathcal{A}} \mathbb{Z}$.

Remark. It follows from the equivariant Hopf type result [33] that

$$\text{mDeg}\, f_i - \text{mDeg}\, f_{0,i} \in \oplus_{H \in \mathcal{A}} |D_i| \mathbb{Z}.$$

In addition, the following extendability result is known.

Proposition 5.9 ([33])**.** *Let $N = S^1 \times_D DV$ be as before and $f : \partial N \to SW_{\text{free}}$ be an S^1-map. Set* $\text{mDeg}\, f = (d_H(f))$.

(1) *$f : \partial N \to SW_{\text{free}}$ is extendable to an S^1-isovariant map $\tilde{f} : N \to SW$ if and only if $d_H(f) = 0$ for any $H \in \mathcal{A}$ with $H \not\leq D$.*

(2) *For any extendable f and for any $(a_H) \in \oplus_{H \in \mathcal{A}} |D| \mathbb{Z}$ satisfying $a_H = 0$ for $H \in \mathcal{A}$ with $H \not\leq D$, there exists an S^1-map $f' : \partial N \to SW_{\text{free}}$ such that f' is extendable to an S^1-isovariant map $\tilde{f}' : N \to SW$ and* $\text{mDeg}\, f' = \text{mDeg}\, f + (a_H)$.

Using these propositions, one can see that there are S^1-isovariant maps $f_i : \partial N_i \to SW$ such that $\coprod_i f_i$ extends both on X and on $\coprod_i N_i$ as isovariant maps. Thus an isovariant map from M to SW is constructed.

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Representation spaces fulfilling the gap hypothesis

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ABSTRACT. For a finite group G, an $\mathcal{L}(G)$-free gap G-module V is a finite dimensional real G-representation space satisfying the two conditions: (1) $V^L = 0$ for any normal subgroup L of G with prime power index, (2) $\dim V^P > 2 \dim V^H$ for any $P < H \leq G$ such that P is a subgroup of prime power order. A finite group G, not of prime power order, is called a gap group if there is an $\mathcal{L}(G)$-free gap G-module. In this paper, we survey techniques to construct an $\mathcal{L}(G)$-free gap G-module and review gap groups; in addition, we announce a new family of gap groups.

1 Introduction

For a manifold M with an action of a finite group G, M is said to satisfy the *gap hypothesis* for a pair (P, H) of subgroups of G if

$$\dim M^P > 2 \dim M^H$$

holds. The gap hypothesis for (P, H) enables us to deform continuous maps $S^k \to M^P$, where $k \leq \dim M^P/2$, to maps $S^k \to M^P \smallsetminus M^H$ by homotopies in M^P. Many researchers have studied equivariant surgery theories under the gap hypothesis or stronger hypotheses (cf. [Pet78, DP83, MR84, PR84, Wan84, DS87, DR88, Ste88, Mor89, DS90, LM90, RT91, CW92, Sch92, Wei99, Sch06, Web07]).

Let G be a finite group and p a prime. In this paper we regard the trivial group as a p-group. We denote by $\mathcal{P}_p(G)$ the set of p-subgroups of G, by $O^p(G)$, called the *Dress subgroup* of type p, the smallest normal subgroup of G whose index is a power of p, possibly 1, and by $\mathcal{L}_p(G)$ the family of subgroups L of G which contains the

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Dress subgroup $O^p(G)$ of type p. We denote by $\pi(G)$ the set of prime divisors of the order of G. Set

$$\mathcal{P}(G) = \bigcup_{p \in \pi(G)} \mathcal{P}_p(G) \quad \text{and} \quad \mathcal{L}(G) = \bigcup_{p \in \pi(G)} \mathcal{L}_p(G).$$

We call a finite dimensional real G-representation space a *G-module*. Let V be a G-module. For a set $\mathcal{F}$ of subgroups of G, we say that V is *$\mathcal{F}$-free* if $V^H = \{0\}$ for all $H \in \mathcal{F}$. An $\mathcal{L}(G)$-free G-module V is called a *gap G-module* if

$$\dim V^P > 2 \dim V^H$$

for all pairs (P, H) of subgroups of G with $P \in \mathcal{P}(G)$ and $P < H$. A finite group G, not of prime power order, is called a *gap group* if there is a gap G-module. The existence of a gap module played a crucial role in [Pet79, Pet82, Pet83, PR84, CS85, LMP95, LP99, PS02, MP03] in constructing smooth actions on spheres with prescribed fixed point sets. Non-trivial finite perfect groups and finite groups G of odd order with $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ are gap groups. Morimoto and Laitinen [LM98] claimed that finite groups G with both $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and $O^2(G) = G$ are gap groups. So, we are interested in groups with $O^2(G) < G$. Dovermann and Herzog [DH97] showed that symmetric groups S_n, $n \geq 6$, are gap groups by specifically using the representation theory of the symmetric groups. On the other hand, Morimoto and Yanagihara [MY96] showed that S_5 is not a gap group. Morimoto, Sumi and Yanagihara [MSY00] studied the construction of a gap module by elementary techniques and showed that $S_5 \times S_4$ and $S_n \times C_2$ are gap groups for $n \geq 6$, where C_2 is a cyclic group of order 2. Sumi extended their techniques and obtained a necessary and sufficient condition for a group to be a gap group.

In this paper, we survey their constructions of gap modules and review results obtained so far on gap groups and non-gap groups. Furthermore we announce a new family of gap groups.

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2 The module $V(G)$

Let G be a finite group, not of prime power order. If $\mathcal{P}(G) \cap \mathcal{L}(G) \neq \varnothing$, then there is no gap G-module, since putting $P \in \mathcal{P}(G) \cap \mathcal{L}(G)$ and $H = G$, we have $P < H$ and

$$\dim V^P = 2 \dim V^H = 0$$

for an arbitrary $\mathcal{L}(G)$-free G-module V. Therefore every finite group G, not of prime power order, with $\mathcal{P}(G) \cap \mathcal{L}(G) \neq \varnothing$ has no gap G-module. So, from now on, we assume that $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$.

We denote by $\mathcal{D}(G)$ the set of all pairs (P, H) of subgroups of G such that $P < H \leq G$ and $P \in \mathcal{P}(G)$. For a G-module V, we define a function $d_V \colon \mathcal{D}(G) \to \mathbb{Z}$ by

$$d_V(P, H) = \dim V^P - 2 \dim V^H.$$

We say that V is *positive* (resp. *non-negative*) at (P, H) if $d_V(P, H)$ is positive (resp. non-negative), and that V is *positive* (resp. *non-negative*) on $\mathcal{E}$ if V is positive (resp. non-negative) at any element of $\mathcal{E}$ for a subset $\mathcal{E}$ of $\mathcal{D}(G)$. Then an $\mathcal{L}(G)$-free G-module V is a gap module if and only if V is positive on $\mathcal{D}(G)$.

For a finite group G, we define the G-module

$$V(G) = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \in \pi(G)} (\mathbb{R}[G/O^p(G)] - \mathbb{R}).$$

If P is a group of prime power order, then $V(P) = \{0\}$ holds. Laitinen and Morimoto [LM98] show that $V(G)$ is an $\mathcal{L}(G)$-free G-module which is non-negative on $\mathcal{D}(G)$. Furthermore they obtained

Theorem 2.1 ([LM98, MSY00]). *Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$. $V(G)$ is a gap G-module if G satisfies $O^2(G) = G$, or $O^p(G) \neq G$ for at least two odd primes p.*

As a corollary, $V(G)$ is a gap G-module if G is a non-trivial perfect group or a group of odd order (with $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$).

In general $V(G)$ is not a gap G-module even if G is a gap group.

Theorem 2.2 ([MY96, DH97]). *The symmetric group S_n on n letters is a gap group if and only if $n \geq 6$.*

Note that $V(S_n)$ is not a gap S_n-module. Dovermann and Herzog specifically used the representation theory of the symmetric groups, so it seems difficult to use their techniques for general groups. Morimoto, Sumi and Yanagihara [MSY00] gave a construction of a gap module by assembling induced modules for certain subgroups.

We set

$$\mathcal{D}^2(G) = \{(P, H) \in \mathcal{D}(G) \mid [H : P] = 2 = [O^2(G)H : O^2(G)P], \\ O^q(G)P = G \text{ for all odd prime } q\}.$$

If we have an $\mathcal{L}(G)$-free G-module W which is positive on $\mathcal{D}^2(G)$ then

$$W \oplus (\dim W + 1)V(G)$$

is an $\mathcal{L}(G)$-free gap G-module. If G is an extension of a gap group N by a group of odd order then $\mathrm{Ind}_N^G W$ is positive on $\mathcal{D}^2(G)$ for a gap N-module W, and thus G is a gap group ([MSY00, Lemma 0.3]).

For a G-module V, we define the G-module $V_{\mathcal{L}(G)}$ as the orthogonal complement of the subspace

$$\mathrm{Span}(v \in V \mid Lv = v \text{ for some } L \in \mathcal{L}(G))$$

in V with respect to a G-invariant inner product. Then $V_{\mathcal{L}(G)}$ is the maximal $\mathcal{L}(G)$-free G-submodule V. We put

$$V(C; G) = (\mathrm{Ind}_C^G(\mathbb{R}[C] - \mathbb{R}))_{\mathcal{L}(G)}$$

for a subgroup C of G. However $V(C; G)$ is not necessarily non-negative on $D^2(G)$.

Theorem 2.3 ([MSY00])**.** *The groups $S_4 \times C_m$ and $S_5 \times C_m$ with $m \geq 3$ divisible by some odd prime are gap groups, as well as S_n and $S_n \times C_2$ with $n \geq 6$.*

For $G = S_n \times C_2$, $n \geq 7$, Morimoto, Sumi and Yanagihara constructed a gap G-module by using G-modules $V(G)$, $V(C; G)$ for some cyclic subgroup of G, and the induced G-module from a gap $(S_{n-1} \times C_2)$-module.

3 Non-gap groups

In the previous section we saw that there are infinitely many isomorphism classes of gap groups. A finite group G is said to be a non-gap group if G is of prime power order or G is not a gap group. In this section we concentrate our attention on non-gap groups.

Theorem 3.1 ([MSY00, Proposition 3.1])**.** *A group G which has a non-gap subgroup of G with index a power of 2 is a non-gap group.*

For example, $S_5 \times K$ is a non-gap group for a 2-group K. Thus, there are infinitely many isomorphism classes of non-gap groups, not of prime power order. Note that $S_5 \times K$ is a gap group for a cyclic group K, not of 2-power order. So it seems that non-gap groups are relatively uncommon.

Theorem 3.2 ([Sum01, Corollary 6.7])**.** *Let m be a non-negative integer and suppose that $n_1, \ldots, n_m$ are positive integers. Let K be a direct product $\prod_{j=1}^{m} D_{2n_j}$ of dihedral groups D_{2n_j} of order $2n_j$ and N a p-group, a direct product of copies of S_4, or S_5. Then $N \times K$ is a non-gap group.*

Thus for a direct product of symmetric groups we have the following.

Theorem 3.3 ([Sum01, Theorem 6.8])**.** *Let m be a positive integer, and $n_1, \ldots, n_m$ integers such that $n_1 \geq n_2 \geq \cdots \geq n_m > 1$. The direct product group of symmetric groups $S_{n_1}, \ldots, S_{n_m}$ is a non-gap group if and only if*

(1) $n_1 \leq 4$, *or*

(2) $m = 1$ *and* $n_1 = 5$, *or*

(3) $m \geq 2$, $n_1 = 5$ *and* $n_2 \leq 3$.

To show that a finite group G is a gap group it suffices to construct a gap G-module. However, we need some techniques to show that it is a non-gap group. We have the following theorem obtained by using a 'dimension matrix' which is introduced in the next section.

Theorem 3.4 ([Sum01, Theorem 1.1])**.** *Let G be a finite group such that $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and let D be a dihedral group. Then G is a non-gap group if and only if $G \times D$ is a non-gap group.*

In Theorem 3.4, the condition $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ is not removable, since a cyclic group C_{15} of order 15 is a non-gap group but $C_{15} \times D_6$ is a gap group.

4 The gap hypothesis

Morimoto, Sumi and Yanagihara obtained

Theorem 4.1 ([MSY00]). *Let G be a finite group such that $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$. If there is an epimorphism $\pi\colon G \to K$ such that $K \times C_2$ is a gap group, then G is a gap group, where C_2 is a group of order 2.*

They also showed that $S_n \times C_2$ is a gap group for $n \geq 6$ and then, as a corollary, obtained that a finite group which has a quotient group isomorphic to S_n for some integer $n \geq 6$ is a gap group. Thus we have a natural question: Is it true that if G is a gap group then so is $G \times C_2$? This question was affirmatively solved in [Sum01]. Thus we have the following.

Theorem 4.2 ([Sum01, Theorem A]). *If a finite group G has a quotient group which is a gap group, then G itself is a gap group.*

Let n be the number of all isomorphism classes of $\mathcal{L}(G)$-free irreducible G-modules and $m = |D(G)|$. We denote by $M(m, n; \mathbb{Z})$ the set of $m \times n$ matrices with entries in $\mathbb{Z}$. We say that D is a *dimension matrix* of G, if $D \in M(m, n; Z)$ is a matrix of the form

$$\begin{pmatrix} d_{V_1}(P_1, H_1) & d_{V_2}(P_1, H_1) & \dots & d_{V_n}(P_1, H_1) \\ d_{V_1}(P_2, H_2) & d_{V_2}(P_2, H_2) & \dots & d_{V_n}(P_2, H_2) \\ \vdots & \vdots & & \vdots \\ d_{V_1}(P_m, H_m) & d_{V_2}(P_m, H_m) & \dots & d_{V_n}(P_m, H_m) \end{pmatrix}$$

where $[V_j]$ is taken over isomorphism classes of $\mathcal{L}(G)$-free irreducible G-modules and (P_i, H_i) is taken over elements of $\mathcal{D}(G)$.

We write $\mathbb{x} \geq \mathbb{y}$ (resp. $\mathbb{x} > \mathbb{y}$) if the inequality holds componentwise: $x_i \geq y_i$ (resp. $x_i > y_i$) for all i, where $\mathbb{x} = {}^t(x_1, \dots, x_n)$ and $\mathbb{y} = {}^t(y_1, \dots, y_n)$. A group G is a gap group if and only if there is $\mathbb{x} \in \mathbb{Z}^n$ such that $\mathbb{x} \geq \mathbb{0}$ and $D\mathbb{x} > \mathbb{0}$ for a dimension matrix D of G.

For a subset $\mathcal{E}$ of $\mathcal{D}(G)$, a submatrix $D_{\mathcal{E}} \in M(|\mathcal{E}|, n; \mathbb{Z})$ of the dimension matrix D of G is called a *dimension submatrix* of G over $\mathcal{E}$. Set

$$Z_{\mathcal{E}}(G) = \left\{ \mathbb{x} \in \mathbb{Z}^{|\mathcal{E}|} \mid {}^t\mathbb{x} D_{\mathcal{E}} = {}^t\mathbb{0},\ \mathbb{x} \geq \mathbb{0},\ \mathbb{x} \neq \mathbb{0} \right\}.$$

For convenience, we set $Z_{\mathcal{E}}(G) = \varnothing$ in the case where $\mathcal{E}$ is empty.

Now we consider the duality theorem for linear programming over the rational number field $\mathbb{Q}$. The following theorem is well-known as Farkas' Lemma.

Theorem 4.3 (cf. [Sum01, Lemma 4.3])**.** *Let A be an $m \times n$ matrix with entries in $\mathbb{Q}$. For $\mathbb{b} \in \mathbb{Q}^m$, we set*

$$X(A, \mathbb{b}) = \{\mathbb{x} \in \mathbb{Q}^n \mid A\mathbb{x} = \mathbb{b},\ \mathbb{x} \geq \mathbb{0}\}$$

and

$$Y(A, \mathbb{b}) = \{\mathbb{y} \in \mathbb{Q}^m \mid {}^t A\mathbb{y} \leq \mathbb{0},\ {}^t\mathbb{b}\mathbb{y} > \mathbb{0}\}.$$

Then either $X(A, \mathbb{b})$ or $Y(A, \mathbb{b})$ is empty but not both.

By Farkas' Lemma we have a necessary and sufficient condition for the existence of a gap module.

Theorem 4.4 ([Sum01, Proposition 4.5, Corollary 4.6 and Theorem 4.8])**.** *Suppose that there is a subset $\mathcal{T} \subseteq \mathcal{D}(G)$ and an $\mathcal{L}(G)$-free G-module W which is non-negative on $\mathcal{D}(G)$ and positive on $\mathcal{T}$. Then G is a gap group if and only if $Z_{\mathcal{D}(G) \smallsetminus \mathcal{T}}(G) = \varnothing$. In particular a finite group G with $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ is a gap group if and only if $Z_{\mathcal{D}^2(G)}(G) = \varnothing$.*

Recall that a gap module means an $\mathcal{L}(G)$-free real G-representation space which is positive on $\mathcal{D}(G)$. For a gap G-module V, the complexification W of V is an $\mathcal{L}(G)$-free complex G-representation space satisfying

$$\dim_{\mathbb{C}} W^P > 2 \dim_{\mathbb{C}} W^H$$

for $(P, H) \in \mathcal{D}(G)$, since $\dim V^K = \dim_{\mathbb{C}} W^K$ for any subgroup K of G. Conversely, if an $\mathcal{L}(G)$-free complex G-representation space W satisfies

$$\dim_{\mathbb{C}} W^P > 2 \dim_{\mathbb{C}} W^H$$

for $(P, H) \in \mathcal{D}(G)$, then the realification of W is a gap G-module. We also call a complex G-representation space a *complex G-module* and denote by the same symbol, $d_W(P, H)$, the difference $\dim_{\mathbb{C}} W^P - 2 \dim_{\mathbb{C}} W^H$, for a complex G-module W.

For a complex G-module U, we put

$$U_{\mathcal{L}(G)} = (U - U^G) - \bigoplus_{p \in \pi(G)} (U - U^G)^{O^p(G)}.$$

This G-module $U_{\mathcal{L}(G)}$ is the maximal $\mathcal{L}(G)$-free complex G-submodule of U. Let $\mathrm{Irr}_{\mathcal{L}(G)}(G)$ be the set consisting of isomorphism classes of $\mathcal{L}(G)$-free irreducible complex G-modules and let $\mathrm{CycInd}^{\mathrm{out}}_{\mathcal{L}(G)}(G)$ be the set consisting of isomorphism classes of complex G-modules $(\mathrm{Ind}_C^G\xi)_{\mathcal{L}(G)}$ for cyclic subgroups C of G with $C \not\leq O^2(G)$ and C-modules ξ. We note that if $O^2(G) = G$ then $\mathrm{CycInd}^{\mathrm{out}}_{\mathcal{L}(G)}(G)$ and $\mathcal{D}^2(G)$ are both empty and recall that if $O^2(G) = G$ and $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ then G is a gap group.

By Artin's theorem [Ser77, §9.2 Corollary], any real G-module can be written in the real representation ring as a linear combination with rational coefficients of realifications of complex G-modules induced from cyclic subgroups of G. Thus we have the following theorem.

Theorem 4.5. *Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and $O^2(G) \neq G$. The following assertions are equivalent.*

(1) *G is a gap group.*

(2) *There exist non-negative integers k_V for $[V] \in \mathrm{Irr}_{\mathcal{L}(G)}(G)$ such that*

$$\sum_{[V]\in \mathrm{Irr}_{\mathcal{L}(G)}(G)} k_V d_V(P, H) > 0$$

for any $(P, H) \in \mathcal{D}^2(G)$.

(3) *There exist rational numbers q_W for $[W] \in \mathrm{CycInd}^{\mathrm{out}}_{\mathcal{L}(G)}(G)$ such that*

$$\sum_{[W]\in \mathrm{CycInd}^{\mathrm{out}}_{\mathcal{L}(G)}(G)} q_W d_W(P, H) > 0$$

for any $(P, H) \in \mathcal{D}^2(G)$.

(4) *There exist integers n_W for $[W] \in \mathrm{CycInd}^{\mathrm{out}}_{\mathcal{L}(G)}(G)$ such that*

$$\sum_{[W]\in \mathrm{CycInd}^{\mathrm{out}}_{\mathcal{L}(G)}(G)} n_W d_W(P, H) > 0$$

for any $(P, H) \in \mathcal{D}^2(G)$.

We can check (3) in Theorem 4.5 by using the character table (cf. [GAP08, CR81, CR87, CR90, Hup98, JL01]). Furthermore, by combining Theorems 4.4 and 4.5 together, we have the following.

Theorem 4.6. *Let G be a finite group such that $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$. Suppose that there is a subset $\mathcal{T} \subseteq \mathcal{D}^2(G)$ and an $\mathcal{L}(G)$-free G-module which is non-negative on $\mathcal{D}(G)$ and positive on $\mathcal{T}$. Let D be a matrix with entries $d_W(P, H)$, where $[W]$ ranges over $\mathrm{CycInd}^{\mathrm{out}}_{\mathcal{L}(G)}(G)$ corresponding to the columns and (P, H) ranges over $\mathcal{D}^2(G) \smallsetminus \mathcal{T}$ corresponding to the rows. Then G is a gap group if and only if ${}^t\mathrm{y}D = {}^t\mathbb{0}$, $\mathrm{y} \geq \mathbb{0}$ has $\mathrm{y} = \mathbb{0}$ as the only solution.*

5 Non-negative modules

The induced G-module $\mathrm{Ind}_K^G V$ of a K-module V which is non-negative on $\mathcal{D}^2(K)$ is also non-negative on $\mathcal{D}^2(G)$. We can construct a gap G-module by assembling such modules for subgroups K of G and $V(G)$.

Theorem 5.1 (cf. [Sum04, Theorem 3.4])**.** *Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and $O^2(G) < G$. If for each element $h \in G \smallsetminus O^2(G)$ there is a gap subgroup of G containing h then G is a gap group.*

Note that $d_{\mathrm{Ind}_K^G V}(P, H) = 0$ holds for a subgroup K of G, a K-module V positive on $\mathcal{D}^2(K)$, and $(P, H) \in \mathcal{D}^2(G)$ satisfying $PgK \neq HgK$ for any $g \in G$.

For an element x of G, we denote by $\psi(x)$ the set of odd primes q such that there exists a subgroup N of G satisfying $x \in N$ and $O^q(N) \neq N$. We define the subset $E_2(G)$ of $G \smallsetminus O^2(G)$ as the set of elements x of order 2 such that $|\psi(x)| > 1$ or $O^2(C_G(x))$ is not a p-group for any prime p, and define $E_4(G)$ as the set of elements x of $G \smallsetminus O^2(G)$ of order a power of 2 greater than 2 with $|\psi(x)| > 0$. The sets $E_2(G)$ and $E_4(G)$ are invariant subsets of G with respect to the conjugation by elements of G. Set $E(G) = E_2(G) \cup E_4(G)$.

Theorem 5.2 (cf. [Sum04, Propositions 4.1, 4.2, 4.4 and 4.5])**.** *Suppose that $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$. For each $h \in E(G)$, there is an $\mathcal{L}(G)$-free G-module W_h such that W_h is non-negative on $\mathcal{D}(G)$ and positive at $(P, H) \in \mathcal{D}^2(G)$, if $H \smallsetminus P$ meets the conjugacy class (h) of h in G.*

We use the character table (cf. [Ste51]) to show that the projective general linear group $PGL(2, q)$ is a non-gap group for $q = 5, 7, 9, 17$. In case $PGL(2, q)$ is a gap group, we can construct a gap $PGL(n, q)$-module, $n \geq 3$, by assembling induced

$PGL(n,q)$-modules of $\mathcal{L}(K)$-free K-modules which are non-negative on $\mathcal{D}^2(K)$ for subgroups K of $PGL(n,q)$ and just one induced $PGL(n,q)$-module $V(C; PGL(n,q))$ for a cyclic subgroup C.

Theorem 5.3 (cf. [Sum02], [Sum04, Corollary 3.5]). *The projective general linear group $PGL(n,q)$ is a gap group if and only if $n > 2$ or $n = 2$ and $q \neq 2, 3, 5, 7, 9, 17$. The general linear group $GL(n,q)$ is a gap group if and only if $(n,q) \neq (2,2)$, $(2,3)$.*

Note that $GL(2,2)$ is isomorphic to the dihedral group D_6 of order 6 and the order of the centralizer $C_{GL(2,3)}(h)$ is a power of 2 for an element h of $GL(2,3) \smallsetminus SL(2,3)$ of order a power of 2. Putting this together with Theorem 3.2 and [Sum01, Proposition 6.5], we have the following two theorems.

Theorem 5.4. *Suppose that $r \geq 1$ and $n_1, q_1, \ldots, n_r, q_r \geq 2$. The direct product group*

$$\prod_{i=1}^{r} GL(n_i, q_i)$$

is a gap group if and only if there exists an integer i such that $(n_i, q_i) \neq (2,2)$, $(2,3)$.

Theorem 5.5. *Suppose that $r, n_1, q_1, \ldots, n_r, q_r \geq 2$. Then the direct product group*

$$\prod_{i=1}^{r} PGL(n_i, q_i)$$

is a gap group if and only if there is an integer i such that $(n_i, q_i) \neq (2,2)$, $(2,3)$.

All finite simple groups have already been classified (cf. [GLS94, GLS96, GLS98, GLS99, GLS02, GLS05]). There are 26 sporadic groups. Let G be a non-complete sporadic group. Then G is one of twelve sporadic groups: M_{12}, $O'N$, McL, J_3, M_{22}, HS, Suz, Fi_{22}, Fi'_{24}, HN, J_2, and He. Note that $[Aut(G) : O^2(Aut(G))] = 2$, $O^2(Aut(G)) \cong G \cong Inn(G)$ (cf. [CCN+85]).

Theorem 5.6 (cf. [Sum04, Corollary 3.6]). *The automorphism group of a sporadic group is a gap group.*

6 Groups possessing a quotient group of odd prime order

Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$. Recall that if $O^p(G) \neq G$ for at least two odd primes p, or $O^2(G) = G$, then $V(G)$ is a gap G-module. In this section we announce new characterizations of gap groups such that the Dress subgroup of type p_0 is a proper subgroup for a unique odd prime p_0.

Suppose henceforth that $O^{p_0}(G) \neq G$ for a unique odd prime p_0.

Theorem 6.1. *Suppose that $O^2(G) \neq G$. Then G is a gap group if and only if every subgroup K with $O^2(G) \lhd K \leq G$ and $[K : O^2(G)] = 2$ is a gap group.*

In addition, we suppose $[G : O^2(G)] = 2$.

Note that $E_4(G)$ is the set of elements of $G \smallsetminus O^2(G)$ of order a power of 2 greater than 2 and $E_2(G)$ contains elements g of G outside $O^2(G)$ of order 2 such that $O^2(C_G(g))$ is not a p_0-group. Recall that the trivial group is a p_0-group by our convention.

Let S be the set of all conjugacy classes of elements of G of 2-power order outside $O^2(G) \cup E(G)$ and let $\mu_2(K)$ be the 2-power integer such that $|K|/\mu_2(K)$ is odd for a finite group K. Further we put

$$n(g) = \mu_2(C_G(g))/2$$

for $(g) \in S$. Since $[G : O^2(G)] = 2$, we have

$$\sum_{(g)\in S} n(g)^{-1} \leq 1$$

and the equality does not hold if $E(G)$ is not empty.

Theorem 6.2. *The following assertions are equivalent.*

(1) *G is a gap group.*

(2) *$E(G)$ is not empty.*

(3) $\displaystyle\sum_{(g)\in S} n(g)^{-1} \neq 1.$

(4) *There is an element of G outside $O^2(G)$ of order divisible by 4 or*

$$\sum \frac{2}{|C_G(g)/O^2(C_G(g))|} < 1,$$

where g ranges over a complete set of representatives of all conjugacy classes of elements with order 2 in G.

Let D be a finite group having an abelian subgroup A of index 2 and an element h in $G \smallsetminus A$ of order 2 such that $h^{-1}ah = a^{-1}$ for all $a \in A$. We call such a group D a *generalized dihedral group*.

Theorem 6.2 implies the following corollary.

Corollary 6.3. *Let G_2 be a Sylow 2-subgroup of G. If at least one of the following properties holds, then G is a gap group.*

(1) *There are two elements of G_2 outside $O^2(G)$ which are conjugate in G but not conjugate in G_2.*

(2) *G_2 is not a generalized dihedral group.*

Finally we state

Theorem 6.4. *Let K be a finite group such that $\mathcal{P}(K) \cap \mathcal{L}(K) = \varnothing$ and a Sylow 2-subgroup of K is not a generalized dihedral group. If K has a proper normal subgroup with odd index, then K is a gap group.*

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